ALGEBRAIC NUMBER THEORY HW 1: DUE (IN CLASS) WEDNESDAY 2/20/2019

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Question 1. *Show that* $\frac{3+2\sqrt{6}}{1-\sqrt{6}}$ *is an algebraic integer.*

Question 2. *Let A be an integral domain with field of fractions K having characteristic zero. Show that if the ring A is integrally closed, then so is the polynomial ring $A[t]$. Recall that the field of fractions of $A[t]$ is the function field of rational functions*

$$K(t) = \{g(t)/h(t) : \ g, h \in A[t], h \neq 0\},$$

and that $h \mid g$ in $K[t]$ for a separable polynomial h when every root of h in the algebraic closure of K is a root of g.

Question 3. *Show that*

$$54 = 2 \cdot 3 \cdot 3 \cdot 3 = \frac{13+\sqrt{-47}}{2} \cdot \frac{13-\sqrt{-47}}{2}$$

are two distinct factorization of 54 into irreducibles elements of $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{-47})$. Careful: An arbitrary element of $\mathcal{O}_K$ does not have the form $a + b\sqrt{-47}$ neither with $a, b \in \mathbb{Z}$ nor with $a, b \in \frac{1}{2} \cdot \mathbb{Z}$, instead having the form $a + \frac{b}{2}(1 + \sqrt{-47})$ with $a, b \in \mathbb{Z}$. Note $\frac{13+\sqrt{-47}}{2} = 6 + \frac{1}{2}(1 + \sqrt{-47})$ and $\frac{13-\sqrt{-47}}{2} = 7 - \frac{1}{2}(1 + \sqrt{-47})$

Question 4. *If $\mathcal{O}$ is a Dedekind domain and $\mathfrak{A} \subseteq \mathcal{O}$ is a proper, nonzero ideal, show that $\mathfrak{A}^2 \subsetneq \mathfrak{A}$.*

Question 5. *Let $d \in \mathbb{Q}$ with $d > 1$. Show that the only units in the ring $\mathbb{Z}[\sqrt{-d}]$ are ± 1.*