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ON WEIGHTED ZERO-SUM SEQUENCES

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Abstract. Let G be a finite additive abelian group with exponent $\exp(G) = n > 1$ and let A be a nonempty subset of $\{1, \ldots, n-1\}$. In this paper, we investigate the smallest positive integer m, denoted $\mathsf{s}_A(G)$, such that any sequence $\{c_i\}_{i=1}^m$ with terms from G has a length $n = \exp(G)$ subsequence $\{c_{i_j}\}_{j=1}^n$ for which there are $a_1, \ldots, a_n \in A$ such that $\sum_{j=1}^n a_i c_{i_j} = 0$.

When G is a p-group, A contains no multiples of p and any two distinct elements of A are incongruent mod p, we prove that $\mathsf{s}_A(G) \leqslant \lceil \mathsf{D}(G)/|A| \rceil + \exp(G) - 1$ for $|A| \geqslant (\mathsf{D}(G) - 1)/(\exp(G) - 1)$, where $\mathsf{D}(G)$ is the Davenport constant of G and this upper bound for $\mathsf{s}_A(G)$ in terms of $|A|$ is essentially best possible.

In the case $A = \{\pm 1\}$, we determine the asymptotic behavior of $\mathsf{s}_{\{\pm1\}}(G)$ when $\exp(G)$ is even, showing that, for finite abelian groups of even exponent and fixed rank,

$$\mathsf{s}_{\{\pm1\}}(G) = \exp(G) + \log_2 |G| + O(\log_2 \log_2 |G|) \ \text{ as } \ \exp(G) \to \infty.$$

Combined with a lower bound of $\exp(G) + \sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor$, where $G \cong \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ with $1 < n_1 | \cdots | n_r$, this determines $\mathsf{s}_{\{\pm1\}}(G)$, for even exponent groups, up to a small order error term. Our method makes use of the theory of L-intersecting set systems.

Some additional more specific values and results related to $\mathsf{s}_{\{\pm1\}}(G)$ are also computed.

1. Introduction

Let G be a finite abelian group written additively and let $\mathcal{F}(G)$ be the set of all finite, ordered sequences with terms from G, though the ordering will not be of relevance to our investigations apart from notational concerns. A sequence $S = \{c_i\}_{i=1}^n \in \mathcal{F}(G)$ is said to be a zero-sum sequence if $\sigma(S) := c_1 + \cdots + c_n = 0$. In the theory of zero-sums, the constant $\mathsf{s}(G)$ is defined to be the smallest positive integer n such that any sequence of length n contains a zero-sum subsequence of length $\exp(G)$ (the exponent of G). By [10, Theorem 6.2], we have $\mathsf{s}(G) \leq |G| + \exp(G) - 1$. For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ denote the ring of residue classes modulo n. The famous Erdős-Ginzburg-Ziv Theorem (EGZ) [11] (see also [10] and [16]) implies $\mathsf{s}(\mathbb{Z}_n) = 2n - 1$, and the Kemnitz-Reiher Theorem [20] states that $\mathsf{s}(\mathbb{Z}_n^2) = 4n - 3$.

Shortly after the confirmation of Caro's weighted EGZ conjecture [13], which introduced the idea of considering certain weighted subsequence sums, Adhikari and his collaborators (cf. [5] [6] [7]) initiated the study of a new kind of weighted zero-sum problem. Let A be a nonempty subset of $[1, \exp(G) - 1] = \{1, \ldots, \exp(G) - 1\}$. For a sequence $\{c_i\}_{i=1}^n \in \mathcal{F}(G)$, if there are $a_1, \ldots, a_n \in A$ such that $\sum_{i=1}^n a_i c_i = 0$, then the sequence is said to have 0 as an A-weighted

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sum or, simply, to be an A-weighted zero-sum subsequence. Similar to the classical case with $A = \{1\}$, various A-weighted constants can be defined as follows:

- $\mathsf{D}_A(G)$ is the least integer n such that any $S \in \mathcal{F}(G)$ of length $|S| \geq n$ contains a nonempty A-weighted zero-sum subsequence.
- $\mathsf{E}_A(G)$ is the least integer n such that any $S \in \mathcal{F}(G)$ with length $|S| \geq n$ has an A-weighted zero-sum subsequence of length $|G|$.
- $\mathsf{s}_A(G)$ is the least integer n such that any $S \in \mathcal{F}(G)$ with length $|S| \geq n$ has an A-weighted zero-sum subsequence of length $\exp(G)$.

The conjecture that $\mathsf{E}_A(G) = |G| + \mathsf{D}_A(G) - 1$ was recently confirmed [14], rendering the independent study of $\mathsf{D}_A(G)$ and $\mathsf{E}_A(G)$ no longer necessary. See also [5], [26] and [25] for previous partial results on the conjecture.

Let n and r be positive integers. In [4], Adhikari and his coauthors investigated

$$f_A(n, r) := \mathsf{s}_A(\mathbb{Z}_n^r)$$

and proved that $f_{\{\pm 1\}}(n, 2) = 2n - 1$ if n is odd. If p is a prime, $A \subseteq [1, p-1]$, and $\{a \bmod p : a \in A\}$ is a subgroup of the multiplicative group $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$, then the authors in [3] showed that

$$f_A(p, r) \leq \frac{r(p-1)}{|A|} + p \quad \text{for } 1 \leq r < \frac{p|A|}{p-1};$$

in particular, $f_A(p, |A|) \leq 2p - 1$ for such A.

In the present paper, we obtain an essentially sharp upper bound for $\mathsf{s}_A(G)$—without the restriction that $\{a \bmod p : \ a \in A\}$ forms a subgroup of $\mathbb{Z}_p^*$—which is valid for an arbitrary abelian p-group G.

For an abelian p-group $G \cong \mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_r}}$ with $k_1, \ldots, k_r \in \mathbb{N}$, Olson [17] proved that the Davenport constant $\mathsf{D}(G) = \mathsf{D}_{\{1\}}(G)$ equals $\mathsf{d}^*(G) + 1$, where

$$\mathsf{d}^*(G) := \sum_{t=1}^{r} (p^{k_t} - 1).$$

Our first main theorem is the following.

Theorem 1.1. *Let p be a prime and let G be an abelian p-group with $|G| > 1$. Let $\emptyset \neq A \subseteq [1, p^{k_r} - 1] \setminus p\mathbb{Z}$ and suppose that any two distinct elements of A are incongruent modulo p. Then, for each $k \in \mathbb{Z}^+$, any sequence in $\mathcal{F}(G)$ of length at least $p^k - 1 + \lceil (d^*(G) + 1)/|A| \rceil$ contains a nonempty A-weighted zero-sum subsequence whose length is divisible by p^k. Thus, if $|A|\ (\exp(G) - 1) \geq d^*(G) = \mathsf{D}(G) - 1$ (which happens if $|A|$ is at least $\mathsf{rk}\,(G) = r$, the rank of G), then we have*

$$\mathsf{s}_A(G) \leq \exp(G) - 1 + \left\lceil \frac{\mathsf{D}(G)}{|A|} \right\rceil.$$

For any abelian p-group G, our upper bound for $\mathsf{s}_A(G)$ in terms of $|A|$ is essentially best possible, as illustrated by the following example (see also [3] for the particular case $G = \mathbb{Z}_p^r$).

Note that the condition $|A| \geq (\mathsf{D}(G)-1)/(\exp(G)-1)$ cannot be removed even in the classical case $A=\{1\}$ since it is known that $s(\mathbb{Z}_p^2)=4p-3>\mathsf{D}(\mathbb{Z}_p^2)+\exp(\mathbb{Z}_p^2)-1=3p-2$.

Example. Let p be a prime and let $G \cong \mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_r}}$, where $1 \leq k_1 \leq \cdots \leq k_r$. Set $A=[1,l]$ with $l \leqslant p^{k_r}-1$. Consider a sequence S over G which consists of

$$\begin{aligned}
&(0,0,\ldots,0) \quad \text{used } \; p^{k_r}-1 \text{ times},\\
&(1,0,\ldots,0) \quad \text{used } \left\lfloor \frac{p^{k_1}-1}{l} \right\rfloor \text{ times},\\
&(0,1,\ldots,0) \quad \text{used } \left\lfloor \frac{p^{k_2}-1}{l} \right\rfloor \text{ times},\\
&\vdots\\
&(0,\ldots,0,1) \quad \text{used } \left\lfloor \frac{p^{k_r}-1}{l} \right\rfloor \text{ times}.
\end{aligned}$$

Clearly, S contains no subsequence of length $\exp(G)=p^{k_r}$ which has 0 as an A-weighted sum. Note that the length of S is $p^{k_r}-1+\sum_{t=1}^{r}\lfloor (p^{k_t}-1)/l\rfloor$, which coincides with $p^{k_r}-2+\lceil (d^*(G)+1)/l\rceil=\exp(G)-2+\lceil \mathsf{D}(G)/|A|\rceil$ when l divides every $p^{k_t}-1$, which may easily be arranged, for instance, if all the k_t are equal.

Under the conditions of Theorem 1.1, Thangadurai [24] showed that $\mathsf{D}_A(G) \leqslant \lceil \mathsf{D}(G)/|A|\rceil$ via the group ring method. This result is an easy consequence of Theorem 1.1 since we may add $\exp(G)-1$ 0's to a sequence in $\mathcal{F}(G)$ of length $\lceil \mathsf{D}(G)/|A|\rceil$ and then apply our theorem.

As we have already mentioned, in [4] it was proved that $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_n^2)=2n-1=2\exp(G)-1$ if n is odd. It is easy to see (and is a specific case in the Kemnitz-Reiher Theorem [20] mentioned before) that $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_2^2)=5=2\exp(G)+1$.

In contrast to these results, in this paper we fully determine the asymptotic behavior of $\mathsf{s}_{\{\pm 1\}}(G)$ when $\exp(G)$ is even, showing that, for finite abelian groups of even exponent and fixed rank,

$$\mathsf{s}_{\{\pm 1\}}(G)=\exp(G)+\log_2|G|+O(\log_2\log_2|G|) \;\text{ as }\; \exp(G)\to\infty. \tag{1.1}$$

More precisely, we prove the following theorem.

Theorem 1.2. *Let $r \geq 1$ be an integer. Then there exists a constant C_r, dependent only on r, such that*

$$\mathsf{s}_{\{\pm 1\}}(G) \leq \exp(G)+\log_2|G|+C_r\log_2\log_2|G|$$

for every finite abelian group G of rank r and even exponent.

In view of the lower bound on $\mathsf{s}_{\{\pm 1\}}(G)$ shown below in Theorem 1.3, Theorem 1.2 determines the value of $\mathsf{s}_{\{\pm 1\}}(G)$ up to the very small order error term given in (1.1). Our method makes use of fundamental results from the theory of L-intersecting set systems and could be used to explicitly estimate the coefficient C_r in specific cases as well as give bounds for how long a sequence $S \in \mathcal{F}(G)$ must be to ensure a $\{\pm 1\}$-weighted zero-sum subsequence of even length n, where n is any even integer at least $(r+1)2^{r+1}$ and $r = \mathsf{rk}\,(G)$. To illustrate this point, and to gently accustom the reader to the method in a more concrete setting, we first calculate some specific values of $\mathsf{s}_{\{\pm 1\}}(G)$ for small $|G|$, and as a by-product of this investigation, obtain the following bounds on the weighted Davenport constant in the case $A = \{\pm 1\}$. Note that

$$\lfloor \log_2 |G| \rfloor = \lfloor \log_2 (n_1 n_2 \cdots n_r) \rfloor = \left\lfloor \sum_{i=1}^{r} \log_2 n_i \right\rfloor,$$

so that the difference between the upper and lower bounds given below is at most r. In the case of cyclic groups, i.e., rank $r = 1$, and 2-groups, this means that equality holds. For the cyclic case, this was first shown in [6]. The results obtained for small $|G|$ can be combined with an inductive argument to yield a simpler upper bound for rank 2 groups, which we handle in brevity at the end of Section 4.

Theorem 1.3. *Let G be a finite abelian group with $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, where $1 < n_1 | \ldots | n_r$. Then*

$$\sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor + 1 \leq \mathsf{D}_{\{\pm 1\}}(G) \leq \lfloor \log_2 |G| \rfloor + 1$$

and

$$\mathsf{s}_{\{\pm 1\}}(G) \geq n_r + \mathsf{D}_{\{\pm 1\}}(G) - 1 \geq \exp(G) + \sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor.$$

2. Proof of Theorem 1.1

The following result is well-known. Alternatively, one may simply note that each side of the equation is a polynomial in x of degree less than n with the difference of both polynomials having $x_1, \ldots, x_n$ as n distinct zeros, and now invoking the fundamental theorem of algebra shows both polynomials must be identical.

Lemma 2.1. (Lagrange's interpolation formula) *Let $P(x)$ be a polynomial over the field of complex numbers, and let $x_1, \ldots, x_n$ be n distinct complex numbers. If $\deg P < n$, then*

$$P(x) = \sum_{j=1}^{n} P(x_j) \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}.$$

The next result is a useful lemma due to the third author (which first appeared in a preprint version of [23] dated May 26, 2003). It provides an alternative to the group-ring method.

Lemma 2.2. (Sun [23, Lemma 4.2]) *Let p be a prime and let $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $m \in \mathbb{Z}$ be integers. Then*

$$\binom{m-1}{p^k - 1} \equiv \begin{cases} 1 \pmod{p} & \text{if } p^k \mid m, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

For convenience, for a polynomial $f(x_1,\ldots,x_n)$ over a field, we use $[x_1^{k_1}\cdots x_n^{k_n}]f(x_1,\ldots,x_n)$ to denote the coefficient of the monomial $x_1^{k_1}\cdots x_n^{k_n}$ in $f(x_1,\ldots,x_n)$.

For a prime p and integer $x\in\mathbb{Z}$, we write $\mathsf{v}_p(x)\in\mathbb{Z}$ for the p-adic valuation of x, which is the maximal $n\in\mathbb{N}$ such that $p^n|x$. If $x\in\mathbb{Q}$, say $x=y/z$ with $y,z\in\mathbb{Z}$, then the p-adic valuation of x is just $\mathsf{v}_p(x)=\mathsf{v}_p(y)-\mathsf{v}_p(z)$. By convention, $\mathsf{v}_p(0)=\infty$.

Proof of Theorem 1.1. Suppose that $G\cong\mathbb{Z}_{p^{k_1}}\oplus\cdots\oplus\mathbb{Z}_{p^{k_r}}$ with $1\le k_1\le\cdots\le k_r$. Then $\mathsf{d}^*(G)=\sum_{t=1}^r(p^{k_t}-1)$ and $\exp(G)=p^{k_r}$. Let $\{c_s\}_{s=1}^n\in\mathcal{F}(G)$ with $n=p^k-1+\lceil(\mathsf{d}^*(G)+1)/|A|\rceil$, where $k\in\mathbb{Z}^+$. We may identify each c_s with a vector

$$\langle c_{s1}\bmod p^{k_1},\ldots,c_{sr}\bmod p^{k_r}\rangle\in\mathbb{Z}_{p^{k_1}}\oplus\cdots\oplus\mathbb{Z}_{p^{k_r}},$$

where $c_{s1},\ldots,c_{sr}$ are suitable integers. Set

$$P(x)=\prod_{a\in A}(x-a)\in\mathbb{Z}[x]\quad\text{and}\quad c=\frac{(-1)^{\mathsf{d}^*(G)+p^k-1}}{P(0)^n}$$

and define

$$f(x_1,\ldots,x_n)=\binom{\sum_{i=1}^n P(x_i)-nP(0)-1}{p^k-1}\prod_{t=1}^r\binom{\sum_{s=1}^n c_{st}x_s-1}{p^{k_t}-1}-c\prod_{i=1}^n P(x_i)\in\mathbb{Q}[x_1,\ldots,x_n].$$

Since $n|A|>\mathsf{d}^*(G)+|A|(p^k-1)$, we have

$$[x_1^{|A|}\cdots x_n^{|A|}]f(x_1,\ldots,x_n)=-c\quad\text{and}\quad\deg f=n|A|.$$

Since $A\cap p\mathbb{Z}=\emptyset$ by hypothesis, we have

$$\mathsf{v}_p(c)=0.$$

As $P(x)\in\mathbb{Z}[x]$ is monic, for each $j\in\mathbb{N}$, there are $q_j(x),r_j(x)\in\mathbb{Z}[x]$ such that

$$x^j=xP(x)q_j(x)+r_j(x)\quad\text{and}\quad\deg r_j\leqslant\min\{j,\deg P\}.$$

Note that $\deg(xP(x)q_j(x))=\deg(x^j-r_j(x))\leqslant j$. Write

$$f(x_1,\ldots,x_n)=\sum_{j_1,\ldots,j_n\geqslant0}f_{j_1,\ldots,j_n}\prod_{i=1}^n x_i^{j_i}\in\mathbb{Q}[x_1,\ldots,x_n].$$

Then, as in the proof of Alon's Combinatorial Nullstellensatz [1], we have

$$\begin{aligned}f(x_1,\ldots,x_n)&=\sum_{j_1,\ldots,j_n\ge0}f_{j_1,\ldots,j_n}\prod_{i=1}^n\left(x_iP(x_i)q_{j_i}(x_i)+r_{j_i}(x_i)\right)\\&=\sum_{i=1}^n x_iP(x_i)h_i(x_1,\ldots,x_n)+\bar f(x_1,\ldots,x_n),\end{aligned}$$

where $h_i(x_1,\ldots,x_n)\in\mathbb{Q}[x_1,\ldots,x_n]$ (the h_i here can be variously chosen), $\deg h_i+\deg(x_iP_i(x))\leqslant\deg f$, and $\bar f$ is given by

$$\bar f(x_1,\ldots,x_n)=\sum_{j_1,\ldots,j_n\ge0}f_{j_1,\ldots,j_n}\prod_{i=1}^n r_{j_i}(x_i).$$

Clearly,

$$f(a_1,\ldots,a_n)=\bar f(a_1,\ldots,a_n)\quad \text{for all } a_1,\ldots,a_n\in A',$$

where $A'=A\cup\{0\}$. Recall $\deg f=n|A|$. Thus, as $\deg h_i+\deg(x_iP_i(x))\leqslant \deg f$, it follows that

$$[x_1^{|A|}\cdots x_n^{|A|}]x_iP(x_i)h_i(x_1,\ldots,x_n)=[x_1^{|A|}\cdots x_n^{|A|}]x_i^{|A|+1}h_i(x_1,\ldots,x_n)=0,$$

whence

$$[x_1^{|A|}\cdots x_n^{|A|}]\bar f(x_1,\ldots,x_n)=[x_1^{|A|}\cdots x_n^{|A|}]f(x_1,\ldots,x_n)=-c.$$

Since $\deg r_j(x)\le \deg P(x)$ for all $j\in\mathbb{N}$, the degree of $\bar f$ in x_i does not exceed $\deg P=|A'|-1$ for any $i=1,\ldots,n$. Applying Lagrange's interpolation formula n times, we obtain

$$\begin{aligned}\bar f(x_1,\ldots,x_n)&=\sum_{a_n\in A'}\bar f(x_1,\ldots,x_{n-1},a_n)\prod_{b\in A'\setminus\{a_n\}}\frac{x_n-b}{a_n-b}\\ &=\cdots=\sum_{a_1,\ldots,a_n\in A'}\bar f(a_1,\ldots,a_n)\prod_{j=1}^{n}\prod_{b\in A'\setminus\{a_j\}}\frac{x_j-b}{a_j-b}\\ &=\sum_{a_1,\ldots,a_n\in A'}f(a_1,\ldots,a_n)\prod_{j=1}^{n}\prod_{b\in A'\setminus\{a_j\}}\frac{x_j-b}{a_j-b}.\end{aligned}$$

By hypothesis, $\mathsf{v}_p(a-b)=0$ for distinct $a,\,b\in A'$. Combining this with

$$[x_1^{|A|}\cdots x_n^{|A|}]\bar f(x_1,\ldots,x_n))=-c\ \text{ and }\ \mathsf{v}_p(c)=0,$$

we conclude from the above interpolation formula that there are $a_1,\ldots,a_n\in A'$ such that

$$\mathsf{v}_p(f(a_1,\ldots,a_n))\le 0.$$

Note that $f(0,\ldots,0)=(-1)^{\mathsf{d}^*(G)+p^k-1}-cP(0)^n=0$ with $\mathsf{v}_p(0)=\infty$. So $I=\{i\in[1,n]:a_i\ne 0\}$ is nonempty. For $i\in I$, we must have $a_i\in A$, and hence $P(a_i)=0$. Thus, recalling that $\mathsf{v}_p(f(a_1,\ldots,a_n))\le 0$ and noting that the left hand side of the below equation is an integer valued polynomial, it follows that

$$\binom{\sum_{i\in[1,n]\setminus I}P(0)-nP(0)-1}{p^k-1}\prod_{t=1}^{r}\binom{\sum_{s\in I}a_sc_{st}-1}{p^{k_t}-1}=f(a_1,\ldots,a_n)\not\equiv 0 \pmod p.$$

With the help of Lemma 2.2, we obtain $-|I|P(0)\equiv 0 \pmod{p^k}$ and $\sum_{s\in I}a_sc_{st}\equiv 0 \pmod{p^{k_t}}$ for all $t=1,\ldots,r$. Therefore $\{c_s\}_{s\in I}$ is an A-weighted zero-sum subsequence of $\{c_i\}_{i=1}^n$ and, since $A\cap p\mathbb{Z}=\emptyset$ implies $P(0)\not\equiv 0 \pmod p$, it follows that the length of this subsequence is $|I|\equiv 0 \pmod{p^k}$.

To see the final part of the theorem, take $k=k_r$ and observe that if $(p^k-1)|A|\geqslant \mathsf{d}^*(G)=\mathsf{D}(G)-1$, then

$$n\le p^k-1+\lceil\frac{\mathsf{d}^*(G)+1}{|A|}\rceil\le p^k-1+\frac{\mathsf{d}^*(G)+|A|}{|A|}\le 2p^k-1,$$

and hence $|I|=p^k$. We are done. $\square$

Remark 2.3. In 2003, the third author Sun [21] obtained the following useful polynomial formula: If $f(x_1,\dots,x_n)$ is a polynomial over a ring R with identity, then, for any $J \subseteq [1,n]$ with $|J| \ge \deg f$, we have

$$\sum_{I\subseteq J}(-1)^{|J|-|I|}f([\![1\in I]\!],\dots,[\![n\in I]\!]) = \Big[\prod_{j\in J}x_j\Big]f(x_1,\dots,x_n),$$

where $[\![s\in I]\!]$ takes the value 1 or 0 according as $s\in I$ or not. Trying to extend this result and the Combinatorial Nullstellensatz, Sun's student Hao Pan made the following (unpublished) observation: If $A_1,\dots,A_n$ are finite, nonempty subsets of a field F and $f(x_1,\dots,x_n)\in F[x_1,\dots,x_n]$ with $\deg f \leqslant \sum_{i=1}^n(|A_i|-1)$, then

$$[x_1^{|A_1|-1}\cdots x_n^{|A_n|-1}]f(x_1,\dots,x_n) = \sum_{a_1\in A_1}\cdots\sum_{a_n\in A_n}\frac{f(a_1,\dots,a_n)}{\prod_{i=1}^n\prod_{a\in A_i\setminus\{a_i\}}(a_i-a)}.$$

3. Terminology and Notation

In this section, we introduce some more notation to be used in the remaining part of the paper.

Let G be an abelian group. Then $\mathcal{F}(G)$ denotes all finite, *unordered* sequences (i.e., multisets) of G written *multiplicatively.* We refer to the elements of $\mathcal{F}(G)$ as sequences. To lighten the notation in parts of the paper, we have previously always written sequences with an implicit order in the format $\{g_i\}_{i=1}^l$, where $g_i\in G$. However, some of the remaining arguments in the paper become more cumbersome to describe without more flexible notation, so we henceforth use the *multiplicative* notation popular among algebraists working in the area (see [9] [10]). In particular, a sequence $S\in\mathcal{F}(G)$ will be written in the form

$$S=\prod_{i=1}^l g_i=\prod_{g\in G}g^{\mathsf{v}_g(S)},$$

where $g_i\in G$ are the terms in the sequence and $\mathsf{v}_g(S)\in\mathbb{N}=\{0,1,2,\dots\}$ denotes the multiplicity of the element g in S. Note that the p-adic valuation of an integer x is just the multiplicity of p in the prime factorization of $x=p_1\cdot\ldots\cdot p_l$, which is indeed where the notation originates. Then $|S|=l$ is the length of the sequence, $S'|S$ denotes that S' is a subsequence of S and, in such case, $S'^{-1}S$ denotes the subsequence of S obtained by removing the terms of S' from S. The support of S, denoted $\mathrm{supp}(S)$, consists of all $g\in G$ which occur in S, i.e., all $g\in G$ with $\mathsf{v}_g(S)\ge 1$. Of course, if $S,\,T\in\mathcal{F}(G)$ are two sequences, then $ST\in\mathcal{F}(G)$ denotes the sequence obtained by concatenating S and T. For a homomorphism $\varphi: G\to G'$, we use $\varphi(S)$ to denote the sequence in G' obtained by applying φ to each term of S. Finally, $\sigma(S)=\sum\limits_{i=1}^{l}g_i$ denotes the sum of the terms of the sequence S.

Let $X,\,Y\subseteq G$. Then their sumset is the set

$$X+Y=\{x+y\mid x\in X,\,y\in Y\}$$

and $-X = \{-x \mid x \in X\}$ denotes the set of inverses of X. If $A \subseteq \mathbb{Z}$ and $g \in G$, then

$$A \cdot g = \{ag \mid a \in A\}.$$

We say that $g \in G$ is an A-weighted n-term subsequence sum of $S \in \mathcal{F}(G)$, or simply an A-weighted n-sum of S, if there is an n-term subsequence $g_1 \cdot \ldots \cdot g_n$ of S and $a_i \in A$ such that $g = \sum_{i=1}^{n} a_i g_i$. If we only say g is an A-weighted subsequence sum of $S \in \mathcal{F}(G)$, then we mean it is a A-weighted n-sum of S for some $n \geq 1$. When we say that $S = g_1 \cdot \ldots \cdot g_n \in \mathcal{F}(G)$ has g as an A-weighted sum, this means there are $a_i \in A$ such that $g = \sum_{i=1}^{n} a_i g_i$. A sequence having the element 0 as an A-weighted sum will simply be called an A-weighted zero-sum sequence.

4. Plus-Minus Weighted Zero-Sums: Generic Bounds and Results for Small $|G|$

In this section, we focus on A-weighted subsequence sums when $A = \{\pm 1\}$ and use the multiplicative notation for sequences described in Section 3. We begin with an important observation. Let G be an abelian group, let $S \in \mathcal{F}(G)$ be a sequence, and let S' be a subsequence of S, say $S' = g_1 \cdot \ldots \cdot g_n$ with $g_i \in G$. Then $\sum_{i=1}^{n} A \cdot g_i$ is the set of all A-weighted sums of S'. However, when $A = \{\pm 1\}$, then $A \cdot g_i = A \cdot (-g_i)$, and thus the $\{\pm 1\}$-weighted n-term subsequence sums of S correspond precisely with the those of the sequence $x^{-1}S(-x)$, for $x \in \text{supp}(S)$ and every n. In other words, we can replace any term of the sequence S with its additive inverse without changing which elements of G are A-weighted n-term subsequence sums.

When G is an elementary abelian 2-group, then $x = -x$ for all $x \in G$. Consequently, studying $\{\pm 1\}$-weighted subsequence sums in this case is no different than studying ordinary subsequence sums. In particular (see [18] [20], though the particular cases here are easy to see),

$$\mathsf{D}_{\{\pm 1\}}(\mathbb{Z}_2^2) = \mathsf{D}(\mathbb{Z}_2^2) = 3 \quad \text{and} \quad \mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_2^2) = \mathsf{s}(\mathbb{Z}_2^2) = 5. \tag{4.1}$$

The following Proposition—and the idea behind its proof—will be one of the main tools used for proving the results in this section and the next.

Proposition 4.1. *Let G be a finite and nontrivial abelian group and let $S \in \mathcal{F}(G)$ be a sequence.*

(i) *If $|S| \geq \log_2 |G| + 1$ and G is not an elementary 2-group, then S contains a proper, nontrivial $\{\pm 1\}$-weighted zero-sum subsequence.*

(ii) *If $|S| \geq \log_2 |G| + 2$ and G is not an elementary 2-group of even rank, then S contains a proper, nontrivial $\{\pm 1\}$-weighted zero-sum subsequence of even length.*

(iii) *If $|S| > \log_2 |G|$, then S contains a nontrivial $\{\pm 1\}$-weighted zero-sum subsequence, and if $|S| > \log_2 |G| + 1$, then such a subsequence may be found with even length.*

Proof. We begin with the proof of part (i). Let $S = g_0 \cdot g_1 \cdot \ldots \cdot g_l$, where $g_i \in G$, and set $S' = g_0^{-1} S$. Note

$$l = |S'| = |S| - 1 \geq \log_2 |G| \tag{4.2}$$

by hypothesis. There are 2^l possible subsets $I \subseteq [1,l]$, each of which corresponds to the sequence

$$S'_I := \prod_{i\in I} g_i \in \mathcal{F}(G)$$

obtained by selecting the terms of S' indexed by the elements of I (including the empty selection $I=\emptyset$, corresponding to the trivial/empty sequence, which by definition has sum zero).

Suppose there are distinct subsets $I,\ J \subseteq [1,l]$ with

$$\sigma(S'_I) = \sum_{i\in I} g_i = \sum_{j\in J} g_i = \sigma(S'_J). \tag{4.3}$$

Since $I \setminus J = I \setminus (I \cap J)$ and $J \setminus I = J \setminus (I\cap J)$, we can remove the commonly indexed terms between S'_I and S'_J to find

$$\sigma(S'_{I\setminus J}) = \sum_{i\in I\setminus J} g_i = \sum_{j\in J\setminus I} g_j = \sigma(S'_{J\setminus I}). \tag{4.4}$$

Note, since $I \neq J$, that $I\setminus J$ and $J\setminus I$ cannot both be empty, while $I\setminus J$ and $J \setminus I$ are clearly disjoint. Hence

$$S'_{(I\setminus J)\cup(J\setminus I)} = S'_{I\setminus J}\cdot S'_{J\setminus I} = \prod_{i\in I\setminus J} g_i \cdot \prod_{j\in J\setminus I} g_j$$

is a nontrivial subsequence of S', which, in view of (4.4), has

$$0 = \sum_{i\in I\setminus J} 1\cdot g_i + \sum_{j\in J\setminus I} (-1)\cdot g_j$$

as a $\{\pm 1\}$-weighted sum. Moreover, since $S'_{(I\setminus J)\cup(J\setminus I)}$ is a subsequence of S' with S' being a proper subsequence of S, it follows that $S'_{(I\setminus J)\cup(J\setminus I)}$ is a proper $\{\pm\}$-weighted zero-sum subsequence of S, yielding (i). So we may instead assume there do not exist distinct subsets $I,\ J\subseteq [1,l]$ satisfying (4.3), that is, there are no such subsets with $\sigma(S'_I)=\sigma(S'_J)$.

Now (4.2) implies that there are $2^l \geq |G|$ subsets $I\subseteq[1,l]$. If $2^l > |G|$, then the pigeonhole principle guarantees the existence of distinct subsets satisfying (4.3), contrary to assumption. Therefore we can assume $2^l = |G|$, which is only possible when equality holds in (4.2):

$$|S| = \log_2|G| + 1 \in \mathbb{Z}.$$

Moreover, each of the $2^l = |G|$ subsequences S'_I, where $I\subseteq[1,l]$, must have a distinct sum from G, else the argument from the previous paragraph again completes the proof. In consequence, every element of $G\setminus\{0\}$ is representable as a subsequence sum of S' with 0 represented by the trivial sequence. In particular, it follows that there exist subsequences T_1 and T_2 of S' with $\sigma(T_1) = g_0$ and $\sigma(T_2) = -g_0$, where (recall) g_0 is the term from S that we removed to obtain S'. Consequently, if T_1 is a proper subsequence of S', then g_0T_1 is a proper $\{\pm1\}$-weighted zero-sum subsequence of S, while if T_2 is a proper subsequence of S', then g_0T_2 is a proper $\{\pm1\}$-weighted zero-sum subsequence of S. In either case, the proof of (i) is complete, so we must have $S' = T_2 = T_1$, in which case

$$\sigma(S') = -g_0 = \sigma(T_2) = \sigma(T_1) = g_0. \tag{4.5}$$

Since every element of G occurs as the sum of one of the $|G|$ subsequences S'_I, where $I \subseteq [1, l]$, and since $I = \emptyset$ corresponds to the subsequence with sum 0, we conclude that $\langle \mathrm{supp}(S') \rangle = G$. Consequently, since G is not an elementary 2-group, it follows that there must be some $y \in \mathrm{supp}(S')$ with

$$2y \neq 0. \tag{4.6}$$

Now, recall that replacing a term from a sequence with its additive inverse does not affect any of the $\{\pm 1\}$-weighted subsequence sums (as explained at the beginning of the section). Thus, it suffices to prove (i) for the sequence $S_0 := y^{-1}S(-y)$ obtained by replacing y by $-y$ in S. Note

$$\sigma(S_0) = \sigma(S) - 2y \ \text{ and } \ \sigma(S'_0) = \sigma(S') - 2y,$$

where $S'_0 := y^{-1}S'(-y)$. Therefore, using (4.5), we derive that

$$\sigma(S'_0) = \sigma(S') - 2y = g_0 - 2y. \tag{4.7}$$

However, applying all of the above arguments using $S_0 = y^{-1}S(-y)$ and $S'_0 = y^{-1}S'(-y)$, we will complete the proof unless (4.5) holds for S'_0 as well:

$$\sigma(S'_0) = g_0.$$

Combining this equality with (4.7), we find that $2y = 0$, which contradicts (4.6), completing the proof of (i).

We continue with the proof of part (ii), which is just a variation on that of (i). As before, let $S = g_0 \cdot g_1 \cdot \ldots \cdot g_l$, where $g_i \in G$, and set $S' = g_0^{-1}S$. Note

$$l = |S'| = |S| - 1 \geq \log_2 |G| + 1 \tag{4.8}$$

by hypothesis. By a well-known combinatorial identity (which can be proven using a simple inductive argument and the correspondence between a subset and its compliment), there are

$$2^{l-1} = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{l}{2i} = \sum_{i=0}^{\lfloor (l-1)/2 \rfloor} \binom{l}{2i+1}$$

possible subsets $I \subseteq [1, l]$ of odd cardinality, each of which corresponds to the odd length sequence

$$S'_I := \prod_{i \in I} g_i \in \mathcal{F}(G)$$

obtained by selecting the terms of S' indexed by the elements of I.

Suppose there are distinct subsets $I, J \subseteq [1, l]$ of odd cardinality with

$$\sigma(S'_I) = \sum_{i \in I} g_i = \sum_{j \in J} g_i = \sigma(S'_J). \tag{4.9}$$

Since $I \setminus J = I \setminus (I \cap J)$ and $J \setminus I = J \setminus (I \cap J)$, we can remove the commonly indexed terms between S'_I and S'_J to find

$$\sigma(S'_{I \setminus J}) = \sum_{i \in I \setminus J} g_i = \sum_{j \in J \setminus I} g_j = \sigma(S'_{J \setminus I}). \tag{4.10}$$

Note, since $I \neq J$, that $I \setminus J$ and $J \setminus I$ cannot both be empty, while $I \setminus J$ and $J \setminus I$ are clearly disjoint; furthermore, $|I \setminus J| + |J \setminus I| = |I| + |J| - 2|I \cap J|$ is an even number in view of $|I| \equiv |J| \mod 2$. Hence

$$S'_{(I\setminus J)\cup(J\setminus I)} = S'_{I\setminus J} \cdot S'_{J\setminus I} = \prod_{i\in I\setminus J} g_i \cdot \prod_{j\in J\setminus I} g_j$$

is a nontrivial subsequence of S' with even length, which, in view of (4.10), has

$$0 = \sum_{i\in I\setminus J} 1 \cdot g_i + \sum_{j\in J\setminus I} (-1) \cdot g_j$$

as a $\{\pm 1\}$-weighted sum. Moreover, since $S'_{(I\setminus J)\cup(J\setminus I)}$ is a subsequence of S' with S' being a proper subsequence of S, it follows that $S'_{(I\setminus J)\cup(J\setminus I)}$ is a proper $\{\pm 1\}$-weighted zero-sum subsequence of S, yielding (ii). So we may instead assume there do not exist distinct subsets $I,\ J \subseteq [1, l]$ of odd cardinality satisfying (4.9), that is, there are no such subsets with $\sigma(S'_I) = \sigma(S'_J)$.

Now (4.8) implies that there are $2^{l-1} \geq |G|$ subsets $I \subseteq [1, l]$ of odd cardinality. If $2^{l-1} > |G|$, then the pigeonhole principle guarantees the existence of distinct subsets of odd cardinality satisfying (4.9), contrary to assumption. Therefore we can assume $2^{l-1} = |G|$, which is only possible when equality holds in (4.8):

$$|S'| = \log_2 |G| + 1 \in \mathbb{Z}. \tag{4.11}$$

Moreover, each of the $2^{l-1} = |G|$ odd length subsequences S'_I must have a distinct sum from G, else the argument from the previous paragraph again completes the proof. In consequence, every element of G is representable as an odd length subsequence sum of S'. In particular, it follows that there exist odd length subsequences T_1 and T_2 of S' with $\sigma(T_1) = g_0$ and $\sigma(T_2) = -g_0$, where (recall) g_0 is the term from S that we removed to obtain S'. Consequently, if T_1 is a proper subsequence of S', then g_0T_1 is a proper $\{\pm 1\}$-weighted zero-sum subsequence of S of even length (since the length of T_1 is odd), while if T_2 is a proper subsequence of S', then g_0T_2 is a proper $\{\pm 1\}$-weighted zero-sum subsequence of S of even length (since the length of T_2 is odd). In either case, the proof of (ii) is complete, so we must have $S' = T_2 = T_1$ with $|S'| = |T_1| = |T_2|$ odd, whence

$$\sigma(S') = -g_0 = \sigma(T_2) = \sigma(T_1) = g_0. \tag{4.12}$$

If G is an elementary 2-group, then $\log_2 |G|$ is the rank of G, which is assumed odd by hypothesis. But in this case, (4.11) implies that $|S'| = \log_2 |G| + 1$ is an even number, contrary to what we have just seen above. Therefore we may assume G is not an elementary 2-group.

Since every element of G occurs as the sum of one of the $|G|$ odd length subsequences S'_I of S', we conclude that $\langle \mathrm{supp}(S') \rangle = G$. Consequently, since G is not an elementary 2-group, it follows that there must be some $y \in \mathrm{supp}(S')$ with

$$2y \neq 0. \tag{4.13}$$

The remainder of the proof now concludes identical to that of part (i).

The proof of part (iii) is a routine simplification of the proofs of parts (i) and (ii).□

Next, we give the proof of Theorem 1.3, which is a simple corollary of Proposition 4.1.

Proof of Theorem 1.3. We begin with the first set of inequalities. The upper bound follows from Lemma 4.1(iii). We turn to the lower bound.

Let $e_1, \ldots, e_r$ be a basis for G with $G = \langle e_1\rangle \oplus \langle e_2\rangle \oplus \cdots \oplus \langle e_r\rangle$ and $\langle e_i\rangle \cong \mathbb{Z}_{n_i}$ for $i = 1, \ldots, r$. For $i \in [1, r]$, define

$$S_i = (2^0 e_i) \cdot (2^1 e_i) \cdot (2^2 e_i) \cdot \ldots \cdot (2^{\lfloor \log_2 n_i\rfloor -1} e_i) \in \mathcal{F}(\langle e_i\rangle)$$

and then set

$$S = S_1 S_2 \cdot \ldots \cdot S_r \in \mathcal{F}(G).$$

Note $|S| = \sum_{i=1}^{r} \lfloor \log_2 n_i\rfloor$. Thus it suffices to show S contains no nontrivial $\{\pm1\}$-weighted zero-sum subsequence. Moreover, since the $e_1, \ldots, e_r$ form a basis of G, it in fact suffices to show each S_j, for $j = 1, \ldots, r$, contains no nontrivial $\{\pm1\}$-weighted zero-sum subsequence.

Let $j \in [1, r]$ and consider an arbitrary $\{\pm1\}$-weighted subsequence sum of S_j, say

$$\sum_{i=0}^{\lfloor \log_2 n_j\rfloor -1} \epsilon_i 2^i e_j, \quad \text{where } \epsilon_i \in \{-1, 0, 1\} \ \text{ and not all } \ \epsilon_i = 0.$$

We will show $\sum_{i=0}^{\lfloor \log_2 n_j\rfloor -1} \epsilon_i 2^i e_j \neq 0$. Let $t \in [0, \lfloor \log_2 n_j\rfloor - 1]$ be the maximal index such that $\epsilon_t \neq 0$ and w.l.o.g. assume $\epsilon_t = 1$ (by multiplying all terms by -1 if necessary). Then

$$0 < 1 = 2^t - \sum_{i=0}^{t-1} 2^i \leq 2^t + \sum_{i=0}^{t-1} \epsilon_i 2^i = \sum_{i=0}^{\lfloor \log_2 n_j\rfloor -1} \epsilon_i 2^i \leq \sum_{i=0}^{\lfloor \log_2 n_j\rfloor -1} 2^i \leq n_j - 1,$$

which shows that $\sum_{i=0}^{\lfloor \log_2 n_j\rfloor -1} \epsilon_i 2^i e_j \neq 0$ (since $\mathrm{ord}(e_j) = n_j$). Consequently, S_j contains no $\{\pm1\}$-weighted zero-sum subsequence, for each $j \in [1, r]$, showing that

$$\mathsf{D}_{\{\pm1\}}(G) \geq |S| + 1 = \sum_{i=1}^{r} \lfloor \log_2 n_i\rfloor + 1.$$

To show the second set of inequalities, let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = \mathsf{D}_{\{\pm1\}}(G) - 1$ containing no $\{\pm1\}$-weighted zero-sum subsequence. It is then clear that the sequence $0^{n_r-1}S$ contains no $\{\pm1\}$-weighted zero-sum subsequence of length n_r, showing the first inequality, while the second inequality follows by the first part.□

To show that our method can also be used to precisely determine $\mathsf{s}_{\{\pm1\}}(G)$ in certain cases, we will compute the values of $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2)$ and $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_8^2)$. These will then be used to give a simple bound for $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n^2)$ complementing the result of [4]. We begin with the following lemma.

Lemma 4.2. *Let $G = \mathbb{Z}_8^2$ and let $S \in \mathcal{F}(G)$ be a sequence with $|S| = 10$. Then S contains a $\{\pm1\}$-weighted zero-sum subsequence T of length $|T| \in \{2, 4, 8\}$.*

Proof. We adapt the proof of Lemma 4.1. Let $S = g_1 \cdot g_2 \cdot \ldots \cdot g_{10}$, where $g_i \in G$. For $j \in [0, 10]$, let $\mathcal{I}_j$ be the set of all subsets $I \subseteq [1, 10]$ having cardinality j. We consider $X := \mathcal{I}_4 \cup \mathcal{I}_2$. Recall that we associate each $I \subseteq [0, 10]$ with the indexed subsequence $S_I := \prod_{i\in I} g_i$ of S. We now analyze the possible intersection cardinalities between sets I, $J \in X$ with

$$\sigma(S_I) = \sum_{i\in I} g_i = \sum_{j\in J} g_j = \sigma(S_J). \tag{4.14}$$

Let I, $J \in X$ be distinct indexing subsets such that (4.14) holds. Thus, by removing terms contained in both S_I and S_J, we obtain

$$\sigma(S_{I\setminus J}) = \sum_{i\in I\setminus J} g_i = \sum_{j\in J\setminus I} g_j = \sigma(S_{J\setminus I}).$$

Note, since $I \neq J$, that $I \setminus J$ and $J \setminus I$ cannot both be empty, while clearly $I \setminus J$ and $J \setminus I$ are disjoint. Hence

$$S_{(I\setminus J)\cup(J\setminus I)} = S_{I\setminus J} \cdot S_{J\setminus I} = \prod_{i\in I\setminus J} g_i \cdot \prod_{j\in J\setminus I} g_j \in \mathcal{F}(G)$$

is a nontrivial subsequence of S having

$$0 = \sum_{i\in I\setminus T} 1 \cdot g_i + \sum_{j\in J\setminus T} (-1) \cdot g_j$$

as a $\{\pm 1\}$-weighted zero-sum. Assuming by contradiction that S contains no $\{\pm 1\}$-weighted zero-sum subsequence T of length $|T| \in \{2, 4, 8\}$, we conclude that

$$|S_{(I\setminus J)\cup(J\setminus I)}| = |I \setminus J| + |J \setminus I| = |I| + |J| - 2|I \cap J| \notin \{2, 4, 8\}.$$

Using the above restriction, it is now easily deduced that

$$|I \cap J| = \begin{cases} \text{not possible,} & \text{if } |I| = 2 \text{ and } |J| = 2 \\ 0, & \text{if } |I| = 2 \text{ and } |J| = 4 \\ 1, & \text{if } |I| = 4 \text{ and } |J| = 4 \end{cases} \tag{4.15}$$

for distinct indexing sets I, $J \in X$ satisfying (4.14).

Using (4.15), we proceed to estimate the maximal number of subsets $I \in X$ that can simultaneously have all their corresponding subsequences S_I being of equal sum. Observe these are just very particular L-intersecting set system problems over $|S| = 10$ vertices. To this end, let $I_1, \ldots, I_n \in X$ be distinct indexing subsets with $\sigma(S_{I_j}) = \sigma(S_{I_k})$ for all j and k. We proceed with some useful comments regarding the I_j under this assumption of equal sums.

- In view of (4.15), there can be at most one I_j with $|I_j| = 2$.
- If (say) $|I_1| = |I_2| = |I_3| = 4$ with $|I_1 \cap I_2 \cap I_3| = 1$, then (4.15) implies $|I_1 \cup I_2 \cup I_3| = 10 = |S|$. Thus, since any further I_j with $|I_j| = 2$ must be disjoint from any other I_k (in view of (4.15)), and since $|S| = 10 = |I_1 \cup I_2 \cup I_3|$, we see in this case that no I_j has $|I_j| = 2$. Therefore, if there is a further I_j with $j \geq 4$, then it must have cardinality $|I_j| = 4$, in which case (4.15) shows that I_j can contain at most one element from each set I_1, I_2 and I_3. However, since $|I_1 \cup I_2 \cup I_3| = 10 = |S|$, there are no further elements to be found, whence $|I_j| \leq 3$, a contradiction.

In summary, if three sets I_j of size 4 intersect in a common point, then $n = 3$ and there are no other indexing sets I_j besides these three.

- If (say) $|I_1| = |I_2| = |I_3| = 4$ with $|I_1 \cap I_2 \cap I_3| \neq 1$, then (4.15) ensures that these sets lie as depicted in the following diagram, where each line below represents one of the sets I_j, where $j \in [1, 3]$, with the points contained in the line corresponding to the elements of I_j.

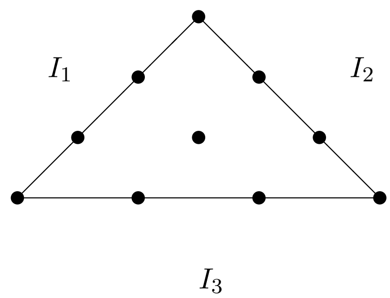

FIGURE 1. Configuration for 3 intersecting 4-sets with no common intersection

Since (4.15) ensures that any I_j with $|I_j| = 2$ must be disjoint from all other I_i, we see there can be no such I_j in this case. Using (4.15) and the previous comment, it is now easily verified that there can be at most two additional I_j with $|I_j| = 4$ besides I_1, I_2 and I_3 (as each of the new points from these additional I_j, for $j \geq 4$, must avoid points already covered by two edges, such as the three corners of the triangle depicted above).

In summary, if there are three I_j of size 4 that do not intersect in a common point, then no I_j has size $|I_j| = 2$ and $n \leq 5$.

- In view of the previous remarks, we see that if some I_j has $|I_j| = 2$, then $n \leq 3$ and all other I_k with $k \neq j$ have $|I_k| = 4$: the first remark ensures that all other I_k have $|I_k| = 4$, while the second and third remark combine to imply there are at most two I_i with $|I_i| = 4$.
- Combining the last three comments, we see that if no I_j has $|I_j| = 2$, that is, $|I_j| = 4$ for all j, then $n \leq 5$.

There are $\binom{10}{4} = 210$ subsets $I \in \mathcal{I}_4$ and $\binom{10}{2} = 45$ subsets $I \in \mathcal{I}_2$. Clearly, many of their corresponding sequences S_I must have common sum as there are only $|G| = 64$ sums to choose from. It is clear, from the final two comments above, that in order to minimize the number of sums spanned by all $I \in X = \mathcal{I}_4 \cup \mathcal{I}_2$, we must pair each $J \in \mathcal{I}_2$ with two I, $I' \in \mathcal{I}_4$ (to form a grouping corresponding to some distinct sum from G) and then take all remaining (unpaired) $I \in \mathcal{I}_4$ and put them into groupings of 5 (with one leftover remainder group possible, and each of these groupings corresponding to some distinct sum from G). In other words, there are at least $45 + \frac{1}{5}(210 - 2 \cdot 45) = 69$ distinct sums covered by the the sets $I \in X$. However, since there are only $|G| = 64 < 69$ sums available, this is a contradiction, completing the proof (essentially, one of the intersection conditions given by (4.15) must actually fail, which then gives rise to the weighted zero-sum subsequence of one of the desired lengths).□

As promised, we now compute the values of $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2)$ and $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_8^2)$ and use them to give a simple upper bound for $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n^2)$. The method below could also be iterated to obtain progressively better bounds for larger $u = \mathsf{v}_2(n)$. However, in view of the results of the next section, we do not expand upon this.

Theorem 4.3. *Let $n \in \mathbb{Z}^+$. Then $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n \oplus \mathbb{Z}_n) \le 2n+1$. Indeed, letting $u = \mathsf{v}_2(n)$ denote the maximum power of 2 dividing n, we have the following bounds:*

(i) $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_2^2) = 5$, $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2) = 8$ *and* $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_8^2) = 14$.
(ii) *If $u \le 1$, then $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n^2) \le 2n+1$.*
(iii) *If $u = 2$, then $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n^2) \le 2n$.*
(iv) *If $u \ge 3$, then $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n^2) \le \frac{15}{8}n+1$.*

Proof of Theorem 4.3. As mentioned in the introduction, if n is odd, then

$$\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_n^2) = 2n-1 \le 2n+1. \tag{4.16}$$

Thus we restrict our attention to the case n even.

We proceed to prove part (i), which contains the crucial basic cases used in the inductive approach. The case $\mathbb{Z}_2^2$ is covered by (4.1), so we begin with $\mathbb{Z}_4^2$.

By Theorem 1.3, we have $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2) \ge 4+2\cdot 2 = 8$. It remains to show $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2) \le 8$. Note $\exp(\mathbb{Z}_4^2) = 4$ and $\log_2|\mathbb{Z}_4^2| = 4$. Let $S \in \mathcal{F}(\mathbb{Z}_4^2)$ be a sequence with length $|S| = 8$. Applying Proposition 4.1(ii) to a subsequence of S of length 6, we obtain a $\{\pm1\}$-weighted zero-sum subsequence T of S with length $|T| \in \{2,4\}$. We may assume $|T| = 2$, else the desired length weighted zero-sum subsequence is found. But now, applying Lemma 4.1(ii) to $T^{-1}S$, we likewise obtain another $\{\pm1\}$-weighted zero-sum subsequence T' of $T^{-1}S$ with length $|T'| = 2$, and then TT' is a $\{\pm1\}$-weighted zero-sum subsequence of S with length 4. This shows $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2) \le 4+2\cdot 2 = 8$

We continue with the case $\mathbb{Z}_8^2$. By Theorem 1.3, we have $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_8^2) \ge 8+2\cdot 3 = 14$. It remains to show $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_8^2) \le 14$. Note $\exp(\mathbb{Z}_8^2) = 8$ and $\log_2|\mathbb{Z}_8^2| = 6$. Let $S \in \mathcal{F}(\mathbb{Z}_8^2)$ be a sequence with length $|S| = 14$ and assume by contradiction that S contains no $\{\pm1\}$-weighted zero-sum subsequence of length 8.

Case 1: There exists a $\{\pm1\}$-weighted zero-sum subsequence S_0 of S with length $|S_0| = 2$. Let R be a maximal length subsequence of S such that $|R|$ is even and, for every $n \in [0,|R|]\cap 2\mathbb{Z}$, R contains a $\{\pm1\}$-weighted zero-sum subsequence T_n of length $|T_n| = n$. In view of the case hypothesis, R exists with $|R| \ge 2$. Note $|R| \le 6$, else R, and hence also S, contains a weighted zero-sum subsequence of length 8, as desired. Thus $|R^{-1}S| = |S| - |R| \ge 14-6 = 8$. Applying Proposition 4.1(ii) to a subsequence of $R^{-1}S$ of length 8, we obtain a $\{\pm1\}$-weighted zero-sum subsequence T of $R^{-1}S$ with length $|T| \in \{2,4,6\}$.

Suppose $|T| \le |R|+2$. Then define $R' = RT$. For every $n \in [2,|R|]\cap 2\mathbb{Z}$, we see that R, and hence also R', contains the $\{\pm1\}$-weighted zero-sum subsequence T_n of length $|T_n| = n$. On the other hand, since $|T|\leq|R|+2$ and $|R|$ is even, it follows that every $m\in[|R|+2,|R|+|T|]\cap 2\mathbb{Z}$ can be written in the form

$$m=|T|+n \text{ with } n\in[|R|-|T|+2,|R|] \text{ and } n\geq 0.$$

Thus, since $|T|$ is even, it follows that the subsequence T_nT of RT is a weighted zero-sum subsequence of length $m\in[|R|+2,|R|+|T|]\cap 2\mathbb{Z}$, in which case RT contradicts the maximality of R. So we conclude that $|T|\geq|R|+4\geq 6$.

Hence, since $|T|\in\{2,4,6\}$, we see that $T|R^{-1}S$ is a weighed zero-sum subsequence of length $|T|=6$. But now, since $|R|\geq 2$, it follows, by the defining property of R, that there exists $\{\pm1\}$-weighted zero-sum subsequence $T_2|R$ with length $|T_2|=2$, whence T_2T is a $\{\pm1\}$-weighted zero-sum subsequence of S with length $|T_2|+|T|=2+6=8$, as desired. This completes the case.

Case 2: There does not exist a $\{\pm1\}$-weighted zero-sum subsequence S_0 of S with length $|S_0|=2$. Since we have assumed by contradiction that S contains no weighted zero-sum subsequence of length 8, and in view of the hypothesis of the case, we see that applying Lemma 4.2 to a subsequence of S of length 10 yields a $\{\pm1\}$-weighted zero-sum subsequence T of S with $|T|=4$. Noting that $|T^{-1}S|=10$, we see that a second application of Lemma 4.2 to $T^{-1}S$ yields another $\{\pm1\}$-weighted zero-sum subsequence $T'|T^{-1}S$ with $|T'|=4$. But now TT' is a $\{\pm1\}$-weighted zero-sum subsequence of S with length $|T|+|T'|=4+4=8$, as desired, which completes the proof of part (i).

Next we prove part (ii). In view of part (i), we have $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_2^2)=5=2n+1$. Thus, in view of (4.16), we may assume $n=2m$ with $m>1$ odd. Let $\varphi:\mathbb{Z}_{2m}^2\to m\cdot\mathbb{Z}_{2m}^2$ denote the multiplication by m homomorphism, which has kernel $\ker\varphi\cong\mathbb{Z}_m^2$ and image $\varphi(\mathbb{Z}_{2m}^2)\cong\mathbb{Z}_2^2$. Note

$$|S|=|\varphi(S)|=2n+1=2(2m-2)+5.$$

Thus, iteratively applying the definition of $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_2^2)=5$ to the sequence $\varphi(S)$, we find $2m-1$ subsequences $S_1,\ldots,S_{2m-1}\in\mathcal{F}(\mathbb{Z}_{2m}^2)$, each of length $|S_i|=2$, such that $S_1S_2\cdot\ldots\cdot S_{2m-1}|S$ and each S_i has a $\{\pm1\}$-weighted sum $x_i\in\ker\varphi\cong\mathbb{Z}_m^2$. Observe, by swapping the signs on every term of S_i, that $-x_i\in\ker\varphi\cong\mathbb{Z}_m^2$ is also a $\{\pm1\}$-weighted sum of S_i. Now applying the definition of $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_m^2)=2m-1$ (see (4.16)) to the sequence $x_1\cdot\ldots\cdot x_{2m-1}$, we find an m-term subsequence, say $x_1\cdot\ldots\cdot x_m$, having 0 as a $\{\pm1\}$-weighted sum. However, since each $\pm x_i$ was a $\{\pm1\}$-weighted sum of the subsequence S_i, we conclude that the subsequence $S_1\cdot\ldots\cdot S_m|S$ has 0 as a $\{\pm1\}$-weighted sum. Since each S_i has length 2, we see $|S_1\cdot\ldots\cdot S_m|=2m=n$. Thus $S_1\cdot\ldots\cdot S_m$ is a $\{\pm1\}$-weighted zero-sum subsequence of S with length n, as desired.

Next, the proof of part (iii). Let $n=4m$ with m odd. By part (i), we know $\mathsf{s}_{\{\pm1\}}(\mathbb{Z}_4^2)=8=2n$. Thus we may assume $m>1$. Let $\varphi:\mathbb{Z}_{4m}^2\to m\cdot\mathbb{Z}_{4m}^2$ denote the multiplication by m homomorphism, which has kernel $\ker\varphi\cong\mathbb{Z}_m^2$ and image $\varphi(\mathbb{Z}_{4m}^2)\cong\mathbb{Z}_4^2$. Note

$$|S|=|\varphi(S)|=2n=4(2m-2)+8.$$

Thus, iteratively applying the definition of $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_4^2) = 8$ to the sequence $\varphi(S)$, we find $2m-1$ subsequences $S_1, \ldots, S_{2m-1} \in \mathcal{F}(\mathbb{Z}_{4m}^2)$, each of length $|S_i| = 4$, such that $S_1 S_2 \cdot \ldots \cdot S_{2m-1} | S$ and each S_i has a $\{\pm 1\}$-weighted sum $x_i \in \ker \varphi \cong \mathbb{Z}_m^2$. Then, as in part (ii), applying the definition of $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_m^2) = 2m-1$ (see (4.16)) to the sequence $x_1 \cdot \ldots \cdot x_{2m-1}$ yields a subsequence (say) $S_1 \cdot \ldots \cdot S_m | S$ which is a $\{\pm 1\}$-weighted zero-sum subsequence of length $|S_1 \cdot \ldots \cdot S_m| = 4m = n$, as desired.

Finally, we conclude with the proof of part (iv). Let $n = 8m$ with $m \in \mathbb{Z}^+$. If $n = 8$, then part (i) implies $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_8^2) = 14 < \frac{15}{8}n + 1 = 16$. We proceed by induction on n. Let $S \in \mathcal{F}(\mathbb{Z}_{8m}^2)$ be a sequence with $|S| \geq \frac{15}{8}n + 1 = 15m + 1$. Let $\varphi : G \to 8 \cdot G$ be the multiplication by 8 map, which has kernel $\ker \varphi \cong \mathbb{Z}_8^2$ and image $\varphi(\mathbb{Z}_{8m}^2) = 8 \cdot \mathbb{Z}_{8m}^2 \cong \mathbb{Z}_m^2$. By induction hypothesis or parts (ii) and (iii), we conclude that $\mathsf{s}_{\{\pm\}}(C_m^2) \leq 2m + 1$. Note

$$|S| = |\varphi(S)| = 13m + (2m + 1).$$

Thus, iteratively applying the definition of $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_m^2) \leq 2m+1$ to the sequence $\varphi(S)$, we find 14 subsequences $S_1, \ldots, S_{2m-1} \in \mathcal{F}(\mathbb{Z}_{8m}^2)$, each of length $|S_i| = m$, such that $S_1 S_2 \cdot \ldots \cdot S_{14} | S$ and each S_i has a $\{\pm 1\}$-weighted sum $x_i \in \ker \varphi \cong \mathbb{Z}_8^2$. Then, as in parts (ii) and (iii), applying the definition of $\mathsf{s}_{\{\pm 1\}}(\mathbb{Z}_8^2) = 14$ (from part (i)) to the sequence $x_1 \cdot \ldots \cdot x_{14}$ yields a subsequence (say) $S_1 \cdot \ldots \cdot S_8 | S$ which is a $\{\pm 1\}$-weighted zero-sum subsequence of length $|S_1 \cdot \ldots \cdot S_8| = 8m = n$, as desired.□

5. Plus-Minus Weighted Zero-Sums: Asymptotic Bounds

For the proof of Theorem 1.2, we will need to make use of several results from the theory of L-interesting set systems. The following is now a well-known result from this area. See [19] for the original t-design formulation, [12] for a more general mod p formulation, and [2] for a yet more general result.

Theorem 5.1 (Uniform Frankl-Ray-Chaudhuri-Wilson Theorem)**.** *Let $k, n \in \mathbb{Z}^+$ be integers, let $\mathcal{F}$ be a collection of k-element subsets of an n-element set, and let $L \subseteq \{0, 1, 2, \ldots, k-1\}$ be a subset. Suppose*

$$|E \cap E'| \in L \quad \text{for all distinct } E, E' \in \mathcal{F}.$$

Then $|\mathcal{F}| \leq \binom{n}{|L|}$.

We will also need a more recent prime power version of the Nonuniform Frankl-Ray-Chaudhuri-Wilson Inequality [8]. To state it, we must first introduce the following definition. We say that a polynomial $f \in \mathbb{Z}[x]$ *separates* the element $\alpha \in \mathbb{Z}$ from the set $B \subseteq \mathbb{Z}$ with respect to the prime p if

$$\mathsf{v}_p(f(\alpha)) < \min_{b \in B} \mathsf{v}_p(f(b)).$$

Theorem 5.2. *Let p be a prime, let $q = p^k$ with $k \geq 1$, and let K and L be disjoint subsets of $\{0, 1, \ldots, q-1\}$. Let $\mathcal{F}$ be a collection of subsets of an n-element set. Suppose*

$$\begin{aligned} |E| &\in K + q\mathbb{Z} \text{ for all } E \in \mathcal{F} \text{ and} \\ |E \cap E'| &\in L + q\mathbb{Z} \text{ for all distinct } E, E' \in \mathcal{F}. \end{aligned}$$

Then $|\mathcal{F}| \leq \binom{n}{D} + \binom{n}{D-1} + \ldots + \binom{n}{0}$, where $D \leq 2^{|L|-1}$ is the maximum over all $\alpha \notin L$ of the minimal degree of a polynomial separating the element α from the set $L + q\mathbb{Z}$ with respect to p.

The following lemma was originally obtained by the third author Z. W. Sun [22, Remark 1.1]. We include the short proof here. For the proof, it will be useful to have the following notation for the number of subsets of an n-element set with size congruent to r modulo m:

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \sum_{\substack{i=0 \\ i \equiv r \mod m}}^{n} \binom{n}{i}.$$

Lemma 5.3 (Sun[22])**.** *Let $m \in \mathbb{Z}^+$. For any $n \in \mathbb{Z}^+$, we have*

$$\begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor \end{bmatrix}_m \geq \frac{2^n}{m}$$

and, furthermore,

$$\begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor \end{bmatrix}_m \geq \begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor + 1 \end{bmatrix}_m \geq \ldots \geq \begin{bmatrix} n \\ \lfloor \frac{n+m}{2} \rfloor \end{bmatrix}_m. \tag{5.1}$$

Proof. By convention, we may set $\binom{n}{i} = 0$ for $i < 0$ and $i > n$, and then the summation in the definition of $\begin{bmatrix} n \\ r \end{bmatrix}_m$ may run over all $i \in \mathbb{Z}$ congruent to r modulo m. Thus, the basic binomial identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, for $n \geq 2$, implies that the same identity holds for the mod m binomial sums:

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \begin{bmatrix} n-1 \\ r \end{bmatrix}_m + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_m, \quad \text{for } n \geq 2. \tag{5.2}$$

Likewise, the other basic binomial identity $\binom{n}{r} = \binom{n}{n-r}$ implies that

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \begin{bmatrix} n \\ n-r \end{bmatrix}_m. \tag{5.3}$$

Together, both these identities, along with a simple proof by induction on n (explained below), can be used to show (5.1).

While the inductive proof is fairly straightforward, it makes use of the fact that the symmetry pairings given by (5.3) do *not* align in the way they do for the usual binomial coefficients. The diagrams in Figure 2 illustrate this point. In all of the diagrams in Figure 2, the bracketed area corresponds to interval of validity for (5.1), the points correspond to values of $\begin{bmatrix} n \\ r \end{bmatrix}_m$, where r varies in the picture, and the points joined by arcs correspond to values of $\begin{bmatrix} n \\ r \end{bmatrix}_m$ which are equal via the relation (5.3). Boxed points are those for which $n - r \equiv n \mod m$, and thus for those points self-symmetric with regards to the relation (5.3).

For $n = 1$ and $n = 2$, the relation (5.1) is clear. This completes the base case. Then, using the relation (5.2), we see that $\begin{bmatrix} n \\ r+1 \end{bmatrix}_m \geq \begin{bmatrix} n \\ r \end{bmatrix}_m$ is equivalent to showing $\begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_m \geq \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_m$. By

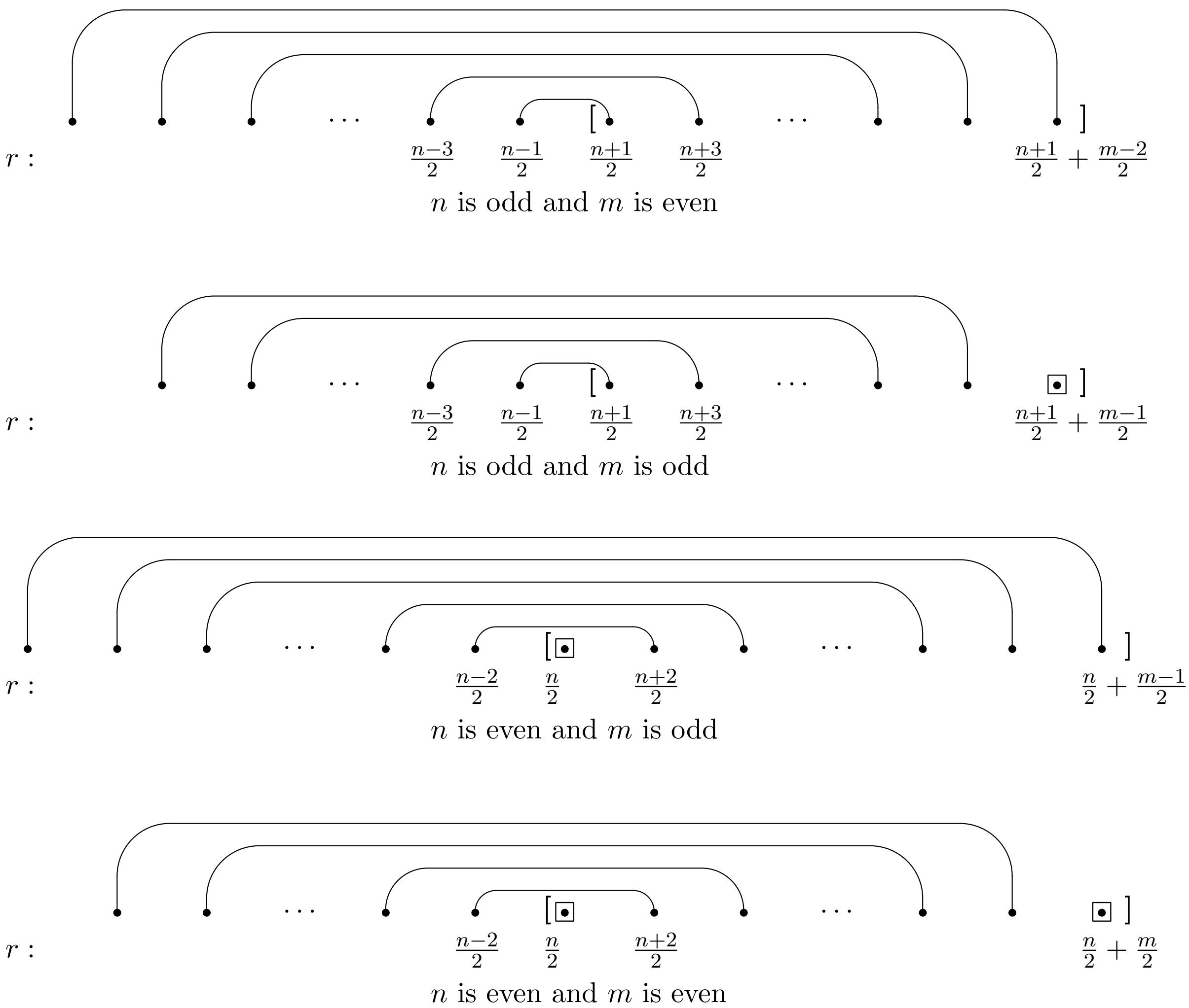

FIGURE 2. Symmetry of $\begin{bmatrix} n \\ r \end{bmatrix}_m$ based on the parity of n and m

induction, this holds so long as $r+1$ and $r-1$ stay within the range for which (5.1) is valid for $n-1$. However, if this is not the case, then either $r+1$ is one too large, in which case (5.3) implies $\begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_m = \begin{bmatrix} n-1 \\ r \end{bmatrix}_m$, or else $r-1$ is one too small, in which case (5.3) implies $\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_m = \begin{bmatrix} n-1 \\ r \end{bmatrix}_m$; see the Figure 3 for the illustration of this point, wherein the parenthesis mark where the old interval of validity of (5.1) using n instead of $n-1$ was. Thus, replacing $\begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_m$ by $\begin{bmatrix} n-1 \\ r \end{bmatrix}_m$ and $\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_m$ by $\begin{bmatrix} n-1 \\ r \end{bmatrix}_m$ as need be, the values now fall within the range of validity for (5.1), whence applying the induction hypothesis completes the proof of (5.1).

However, as is clear from the diagrams in Figure 2, for every r, the range of validity of (5.1) includes either $\begin{bmatrix} n \\ r \end{bmatrix}$ or $\begin{bmatrix} n \\ n-r \end{bmatrix}$. Consequently,

$$\begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor \end{bmatrix}_m = \max_r \begin{bmatrix} n \\ r \end{bmatrix}_m,$$

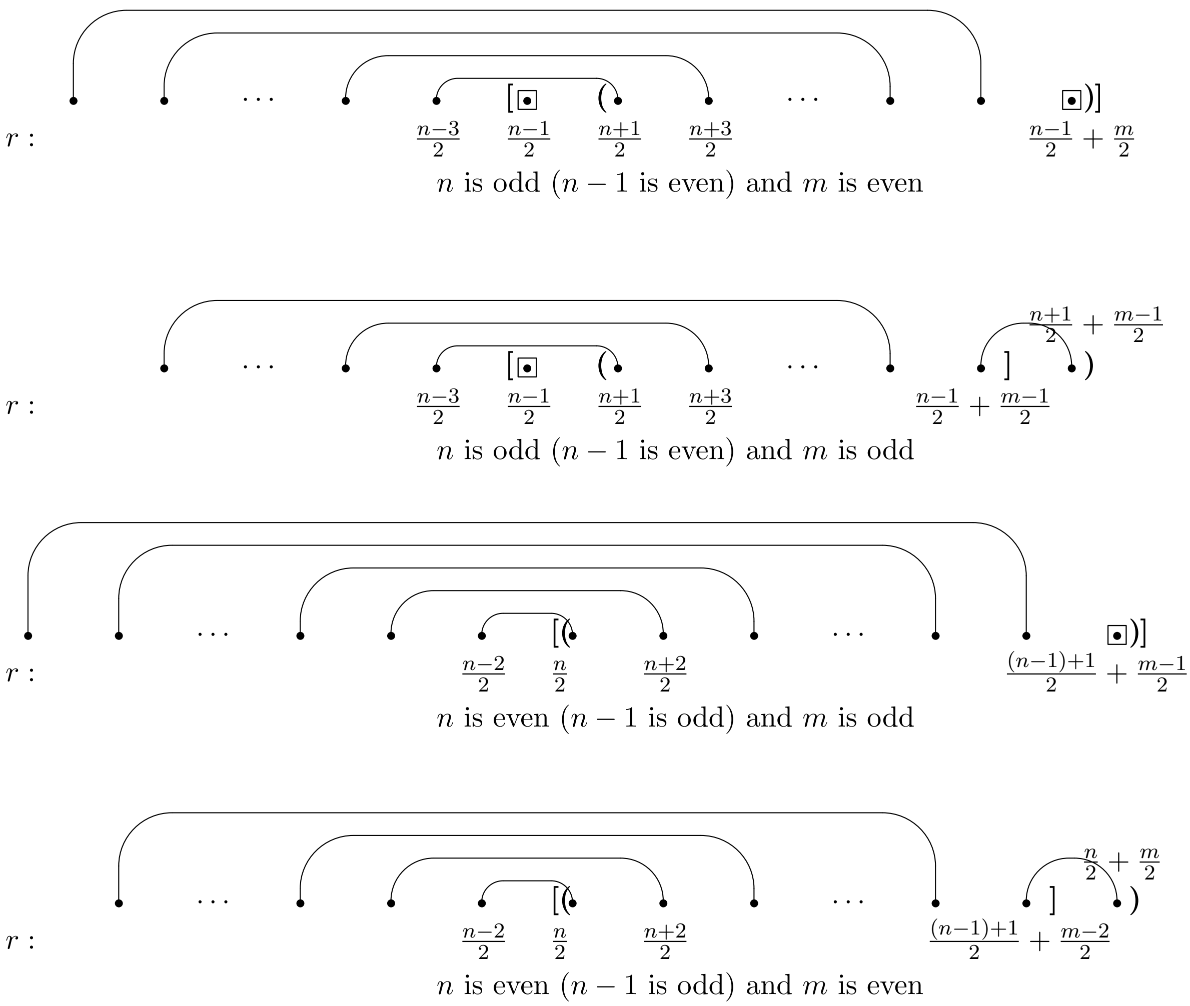

FIGURE 3. Symmetry of $\left[{n-1 \atop r}\right]_m$ based on the parity of n and m

and thus $\left[{n \atop \lfloor \frac{n+1}{2} \rfloor}\right]_m$ is at least the average of the m values $\left[{n \atop r}\right]_m$, which means

$$\left[{n \atop \lfloor \frac{n+1}{2} \rfloor}\right]_m \geq \frac{1}{m}\sum_{r=1}^{m}\left[{n \atop r}\right]_m = \frac{1}{m}\sum_{i=0}^{n}\binom{n}{i} = \frac{2^n}{m},$$

as desired. As mentioned in [22, Remark 1.1], for $m \geq 3$ and $n \geq m-1$, the inductive proof of (5.1) given above can be used to show that all of the inequalities in (5.1) are in fact strict. $\square$

With the above tools in hand, we now conclude with the proof of Theorem 1.2. With regards to asymptotic notation, recall that $f(x) = O(g(x))$ means that there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all sufficiently large values of x, while $f(x) = \Omega(g(x))$ means that there exists a constant $C > 0$ such that $|f(x)| \geq C|g(x)|$ for all sufficiently large values of x, where f and g are functions.

Proof of Theorem 1.2. If $|G| = 2$, then $\log_2 \log_2 |G| = 0$ and $\mathsf{s}_{\{\pm 1\}}(G) = 3 = \exp(G) + \log_2 |G|$. Thus the Theorem holds for any constant C_1, and so we may assume $|G| \geq 4$. In this case, $\log_2 \log_2 |G| \geq 1$, and thus it suffices to prove the existence of C_r when $\exp(G) \geq n_0$ is sufficiently large, as then $\mathsf{s}_{\{\pm 1\}}(G) \leq \exp(G) + \log_2 |G| + C_r \log_2 \log_2 |G| + C'_r$, where $C'_r \geq 0$ is the maximum of $\mathsf{s}_{\{\pm 1\}}(G)$ over all G of rank $\mathsf{rk}\,(G) = r$ and even exponent $\exp(G) < n_0$, and replacing C_r by $C_r + C'_r$ gives the desired constant that works for all G.

The rank $r \geq 1$ will remain fixed throughout the argument. Let G be a finite abelian group of rank r and exponent $n = \exp(G)$ even. Let $m = 2^{r+1}$. Note m depends only on r, and can thus be treated as a constant with regards to asymptotics. We divide the proof into four parts.

Step 1: There exists a constant $C > 0$, dependent only on r, so that any sequence $S \in \mathcal{F}(G)$ with $|S| \geq C n^{r/(r+1)}$ contains a $\{\pm 1\}$-weighted zero-sum subsequence of length m.

First let us see that it suffices to prove that $|S| \geq C' n^{r/(r+1)}$ implies S contains an $\{\pm 1\}$-weighted zero-sum subsequence T of length $|T| \in \{2^1, 2^2, \ldots, 2^{r+1}\}$. Indeed, if we know this to be true, then, for any sequence $S \in \mathcal{F}(G)$ with

$$|S| \geq (C' + (r+1)2^{r+1}) n^{r/(r+1)} \geq C' n^{r/(r+1)} + (r+1)2^{r+1},$$

we can repeatedly apply this result to S to pull off disjoint weighted zero-sum subsequences $T_1, \ldots, T_l$ with $T_1 \cdot \ldots \cdot T_l | S$,

$$|T_1 \cdot \ldots T_l| > (r+1)2^{r+1}, \tag{5.4}$$

and $|T_i| \in \{2^1,\, 2^2, \ldots, 2^{r+1}\}$ for all i. If S contains no such subsequence of length 2^{r+1}, then less than 2^{r+1-j} of the T_i can be of length 2^j, for $j = 1, 2, \ldots, r+1$ (else concatenating a sufficient number of these T_i would yield a weighted zero-sum of the desired length 2^{r+1}). Consequently,

$$|T_1 \cdot \ldots \cdot T_l| < \sum_{j=1}^{r+1} 2^{r+1-j} \cdot 2^j = (r+1)2^{r+1},$$

contradicting (5.4). Thus the step follows with constant $C' + (r+1)2^{r+1}$, and we see it suffices to prove $|S| \geq C' n^{r/(r+1)}$ implies S contains an $\{\pm 1\}$-weighted zero-sum subsequence T of length $|T| \in \{2^1, 2^2, \ldots, 2^{r+1}\}$, as claimed.

Let $S = g_1 \cdot g_2 \cdot \ldots \cdot g_v$, where $g_i \in G$. Let X be the collection of all subsets $I \subseteq [1, v]$ having cardinality 2^r. Recall that we associate each $I \subseteq [1, v]$ with the indexed subsequence $S_I := \prod_{i \in I} g_i$ of S. If $\sigma(S_I) = \sigma(S_J)$ for distinct $I,\, J \in X$, then, by discarding the commonly indexed terms (as we have done several times before in Section 4), we obtain a $\{\pm 1\}$-weighted zero-sum subsequence $S_{I \setminus J} \cdot S_{J \setminus I}$ of S with length

$$|I| + |J| - 2|I \cap J| = m - 2|I \cap J|.$$

Assuming by contradiction that S contains no $\{\pm 1\}$-weighted zero-sum subsequence T with length $|T| \in \{2^1, 2^2, \ldots, 2^{r+1}\}$ and recalling that $m = 2^{r+1}$, we conclude that

$$|I \cap J| \in L := [0, 2^r - 1] \setminus \Big(\frac{m}{2} - \{2^0, 2^1, \ldots, 2^r\} \Big)$$

whenever $\sigma(S_I) = \sigma(S_J)$ with $I, J \in X$ distinct; note $|L| = 2^r - r - 1$. This allows us to give an upper bound on how many distinct $I \in X$ can have equal corresponding sums. Indeed, Theorem 5.1 shows that there can be at most $\binom{v}{|L|} = \binom{v}{2^r-r-1}$ indexing sets from X having equal corresponding sums. Since $|X| = \binom{v}{2^r}$, this implies there are at least

$$\binom{v}{2^r}/\binom{v}{2^r-r-1} = \Omega(v^{r+1})$$

distinct values attained by the $\sigma(S_I)$ with $I \in X$. Since there are at most $|G| \leq \exp(G)^r = n^r$ values in total, we conclude that

$$n^r = \Omega(v^{r+1}),$$

which implies $v < Cn^{r/(r+1)}$ for some constant $C > 0$ (the above asymptotic notation holds for v sufficiently large with respect to r, which is a fixed constant). Thus, if $|S| \geq Cn^{r/(r+1)}$, then S must contain a weighted zero-sum subsequence of one of the desired lengths, completing the step as noted earlier.

Step 2: There exists a constant $C' > 0$, dependent only on r, so that any sequence $S \in \mathcal{F}(G)$ with $|S| \geq C'n^{r/(r+1)}$ contains a $\{\pm 1\}$-weighted zero-sum subsequence T of length $|T| \equiv n \mod m$ and $|T| \leq (r+1)m$.

The proof is a variation on that of Step 1. Let $S = g_1 \cdot g_2 \cdot \ldots \cdot g_v$, where $g_i \in G$. Let $\alpha \in [1, m]$ be the integer such that $n \equiv \alpha \mod m$. Note, since n and m are both even, that α must be an *even* number—this is the only place where the hypothesis regarding the parity of n will be used. Let X be the collection of all subsets $I \subseteq [1, v]$ having cardinality $\frac{1}{2}(\alpha + rm)$, which is an integer as both α and m are even. Recall that we associate each $I \subseteq [1, v]$ with the indexed subsequence $S_I := \prod_{i \in I} g_i$ of S. If $\sigma(S_I) = \sigma(S_J)$ for distinct $I, J \in X$, then, by discarding the commonly indexed terms, we obtain a $\{\pm 1\}$-weighted zero-sum subsequence $S_{I\setminus J} \cdot S_{J\setminus I}$ of S with length

$$|I| + |J| - 2|I \cap J| = \alpha + rm - 2|I \cap J|.$$

Assuming by contradiction that S contains no $\{\pm 1\}$-weighted zero-sum subsequence T with length $\{\alpha, \alpha + m, \alpha + 2m, \ldots, \alpha + rm\}$ and recalling that $m = 2^{r+1}$, we conclude that

$$|I \cap J| \in L := [0, \frac{1}{2}(\alpha + rm) - 1] \setminus \{(r - j)\frac{m}{2} \mid j = 0, 1, \ldots, r\}$$

whenever $\sigma(S_I) = \sigma(S_J)$ with $I, J \in X$ distinct; note $|L| = \frac{1}{2}(\alpha + rm) - r - 1$. This allows us to give an upper bound on how many distinct $I \in X$ can have equal corresponding sums. Indeed, Theorem 5.1 shows that there can be at most $\binom{v}{|L|} = \binom{v}{\frac{1}{2}(\alpha+rm)-r-1}$ indexing sets from X having equal corresponding sums. Since $|X| = \binom{v}{\frac{1}{2}(\alpha+rm)}$, this implies there are at least

$$\binom{v}{\frac{1}{2}(\alpha + rm)}/\binom{v}{\frac{1}{2}(\alpha + rm) - r - 1} = \Omega(v^{r+1})$$

distinct values attained by the $\sigma(S_I)$ with $I \in X$. Since there are at most $|G| \leq \exp(G)^r = n^r$ values in total, we conclude that

$$n^r = \Omega(v^{r+1}),$$

which implies $v < C'n^{r/(r+1)}$ for some constant $C' > 0$. Thus, if $|S| \geq C'n^{r/(r+1)}$, then S must contain a weighted zero-sum subsequence of one of the desired lengths, completing the step.

Step 3: There exists a constant $C'' > 0$, dependent only on r, so that any sequence $S \in \mathcal{F}(G)$ with $|S| \geq \log_2 |G| + C'' \log_2 \log_2 |G|$ contains a $\{\pm 1\}$-weighted zero-sum subsequence T with length $|T| \equiv 0 \mod m$ and $|T| \leq \log_2 |G| + C'' \log_2 \log_2 |G|$.

Suppose we can show that, for any sequence $S \in \mathcal{F}(G)$ with

$$|S| = v \geq \log_2 |G| + C' \log_2 \log_2 |G| \tag{5.5}$$

and

$$v \equiv 0 \mod 2^{r+2} = 2m, \tag{5.6}$$

there is a weighted zero-sum subsequence of S with length congruent to 0 modulo m. Then, since any sequence S with $|S| \geq \log_2 |G| + C' \log_2 \log_2 |G| + 2m$ contains a subsequence S' with

$$\log_2 |G| + C' \log_2 \log_2 |G| \leq |S'| \leq \log_2 |G| + C' \log_2 \log_2 |G| + 2m + 1$$

that also satisfies (5.6), and since any weighted zero-sum subsequence of S' has length trivially bounded from above by $|S'| \leq \log_2 |G| + C' \log_2 \log_2 |G| + 2m + 1$, we see that the step holds setting $C'' = C' + 2m + 1$ (in view of $\log_2 \log_2 |G| \geq 1$). We proceed to show this supposition true.

To that end, let $S = g_1 \cdot g_2 \cdot \ldots \cdot g_v$, where $g_i \in G$, be a sequence satisfying (5.6), in which case $\lfloor \frac{v+1}{2} \rfloor \equiv 0 \mod m$. Let X be the collection of all subsets $I \subseteq [1, v]$ having cardinality $|I| \equiv 0 \mod m$. In view of (5.6) and Lemma 5.3, we find that

$$|X| \geq \frac{2^v}{m} = 2^{v-r-1}. \tag{5.7}$$

Recall that we associate each $I \subseteq [1, v]$ with the indexed subsequence $S_I := \prod_{i \in I} g_i$ of S.

If $\sigma(S_I) = \sigma(S_J)$ for distinct $I,\ J \in X$, then, by discarding the commonly indexed terms, we obtain a $\{\pm 1\}$-weighted zero-sum subsequence $S_{I \setminus J} \cdot S_{J \setminus I}$ of length

$$|I| + |J| - 2|I \cap J|.$$

Hence, since $|I| + |J| \equiv 0 + 0 = 0 \mod m$, we see that $S_{I \setminus J} \cdot S_{J \setminus I}$ will be a $\{\pm 1\}$-weighted zero-sum of length congruent to 0 modulo m provided $|I \cap J| \equiv 0 \mod \frac{m}{2}$. Therefore, assuming to the contrary this is not the case, we conclude that $|I \cap J| \in L$, where $L = \{1, 2, 3, \ldots, 2^r - 1\} + 2^r\mathbb{Z}$, whenever $\sigma(S_I) = \sigma(S_J)$ with $I,\ J \in X$ distinct.

This allows us to give an upper bound on how many distinct $I \in X$ can have equal corresponding sums. Indeed, since all $I \in X$ have $|I| \equiv 0 \mod m = 2^{r+1}$, we see that all $I \in X$ have $I \in K + 2^r\mathbb{Z}$, where $K = \{0\}$. Moreover, the polynomial $f(x) = \prod_{i=1}^{2^r-1}(x - i)$ shows that 0 can be separated from $\{1, 2, 3, \ldots, 2^r - 1\} + 2^r\mathbb{Z}$ with respect to $p = 2$ using a polynomial of degree $D = 2^r - 1$. Thus, applying Theorem 5.2 with $q = p^k = 2^r = \frac{m}{2}$ and using (5.7), we see that there are at least

$$2^{v-r-1} / \left(\sum_{i=0}^{D} \binom{v}{i} \right) = \Omega(2^{v-r-1}/v^{2^r-1})$$

distinct values attained by the $\sigma(S_I)$ with $I \in X$. Hence, since there are at most $|G|$ values in total, we conclude that

$$|G| \geq C\frac{2^{v-r-1}}{v^D} \tag{5.8}$$

for some $C > 0$ when $v \geq v_0$, where $v_0 > 0$ is some constant depending on the fixed constant r (using $D = 2^r - 1$).

Recall, since $|G| \geq 4$, that

$$\log_2 |G| \geq 2 \quad \text{and} \quad \log_2 |G| \geq \log_2 \log_2 |G| \geq 1. \tag{5.9}$$

Let

$$\gamma = D + \max\{0, \log_2(1/C)\} \in \mathbb{Z}^+.$$

Now 2^x is larger than $(x + \gamma + r + 2)^D$ for sufficiently large x. Thus, let $y \geq v_0$ be an integer such that

$$2^x > (x + \gamma + r + 2)^D \quad \text{for all } x \geq y \tag{5.10}$$

and consider $C' = y + \gamma + r + 1$.

Suppose $|S| = v \geq \log_2 |G| + C' \log_2 \log_2 |G|$. Then

$$v = \log_2 |G| + (x + \gamma + r + 1) \log_2 \log_2 |G| \geq v_0$$

for some real number $x \geq y$, and using (5.9), we derive that

$$\begin{aligned} C2^{v-r-1} &= C|G|2^{(x+\gamma+r+1)\log_2\log_2|G|-r-1} \geq C|G|2^{(x+\gamma)\log_2\log_2|G|} \\ &= C|G|(\log_2|G|)^{x+\gamma} = C|G|(\log_2|G|)^D(\log_2|G|)^{x+\max\{0,\log_2(1/C)\}} \\ &\geq C|G|(\log_2|G|)^D 2^{x+\max\{0,\log_2(1/C)\}} \geq 2^x(\log_2|G|)^D|G|, \end{aligned} \tag{5.11}$$

and that, again using (5.9),

$$\begin{aligned} v^D &= (\log_2|G| + (x+\gamma+r+1)\log_2\log_2|G|)^D \\ &\leq ((x+\gamma+r+2)\log_2|G|)^D = (x+\gamma+r+2)^D(\log_2|G|)^D. \end{aligned} \tag{5.12}$$

Combining (5.11) and (5.12) and using (5.10) and $x \geq y$, it follows that

$$C\frac{2^{v-r-1}}{v^D} \geq \frac{2^x(\log_2|G|)^D|G|}{(x+\gamma+r+2)^D(\log_2|G|)^D} = \frac{2^x}{(x+\gamma+r+2)^D}|G| > |G|,$$

contradicting (5.8). Thus we see the constant C' for (5.5) exists, completing the step as remarked earlier.

Step 4: There exists a constant $C_r > 0$, dependent only on r, so that, for sufficiently large n, any sequence $S \in \mathcal{F}(G)$ with $|S| \geq n + \log_2 |G| + C_r \log_2 \log_2 |G|$ contains a $\{\pm 1\}$-weighted zero-sum subsequence T with of length $|T| = n$.

Note that this step will complete the proof, for as remarked at the beginning of the proof, it suffices to prove the theorem for sufficiently large n. For this reason, we may also assume $n \geq (r + 1)m$. Let C, C' and C'' be the respective constants from Steps 1, 2 and 3. We will show that $S \in \mathcal{F}(G)$ contains a $\{\pm 1\}$-weighted zero-sum subsequence of length n provided the length of S satisfies the following three bounds:

$$|S| \geq C' n^{r/(r+1)}, \tag{5.13}$$

$$|S| \geq C n^{r/(r+1)} + (r+1)m + \log_2 |G| + C'' \log_2 \log_2 |G|, \tag{5.14}$$

$$|S| \geq n - m + \log_2 |G| + C'' \log_2 \log_2 |G|. \tag{5.15}$$

Since, for sufficiently large n, the bound given in (5.15) is the maximum of the three bounds, we will subsequently be able to conclude $|S| \geq n - m + \log_2 |G| + C'' \log_2 \log_2 |G|$, for large n, implies S contains a weighted zero-sum subsequence of length n, completing the proof. We continue by showing (5.13)–(5.15) indeed guarantee a length n weighted zero-sum subsequence.

In view of (5.13) and Step 2, we see that S contains a weighted zero-sum subsequence R_0 with

$$|R_0| \equiv n \mod m \text{ and } |R_0| \leq (r+1)m \leq n. \tag{5.16}$$

In view of (5.14), we see that repeated application of Step 1 to $R_0^{-1}S$ yields series of length m weighted zero-sum subsequences, enough so that there exists a subsequence R of $R_0^{-1}S$ with

$$|R| \geq \log_2 |G| + C'' \log_2 \log_2 |G| \text{ and } |R| \equiv 0 \mod m$$

such that, for every $k \in [0, |R|] \cap m\mathbb{Z}$, there is a $\{\pm 1\}$-weighted zero-sum subsequence T_k of R with length $|T_k| = k$. Choose such a subsequence R of $R_0^{-1}S$ with length $|R|$ maximal. Since $|R_0| \equiv n \mod m$ with $|R_0| \leq n$ (in view of (5.16)), we see that $n = |R_0| + ym$ for some $y \in \mathbb{N}$. Thus $|R_0 R| \leq n - m$, else the proof is complete.

In view of $|R_0 R| \leq n - m$ and (5.15), we see that

$$|R_0^{-1} R^{-1} S| \geq \log_2 |G| + C'' \log_2 \log_2 |G|.$$

Hence, applying Step 3 to $R_0^{-1} R^{-1} S$, we find a nontrivial weighted zero-sum subsequence T of $R_0^{-1} R^{-1} S$ with $|T| \equiv 0 \mod m$ and

$$|T| \leq \log_2 |G| + C'' \log_2 \log_2 |G|.$$

We claim that RT contradicts the maximality of $|R|$, which, once shown true, will provide the concluding contradiction for the proof.

Since $|R| \equiv |T| \equiv 0 \mod m$, we have $|RT| \equiv 0 \mod m$, while

$$|TR| \geq |R| \geq \log_2 |G| + C'' \log_2 \log_2 |G|.$$

For every $k \in [0, |R|] \cap m\mathbb{Z}$, the weighted zero-sum subsequence T_k divides R, and hence also RT, and is of length $|T_k| = k$. Since

$$|R| \geq \log_2 |G| + C'' \log_2 \log_2 |G| \geq |T| - m,$$

it follows that every $t \in [|R| + m, |R| + |T|] \cap m\mathbb{Z}$ can be written in the form $t = k + |T|$ with $k \in [|R| - |T| + m, |R|] \cap m\mathbb{Z} \subseteq [0, |R|] \cap m\mathbb{Z}$. Hence the subsequence $T_k T$ of RT is a weighted zero-sum subsequence of length $t \in [|R| + m, |R| + |T|] \cap m\mathbb{Z}$, which shows that the subsequence RT of $R_0^{-1}S$ indeed contradicts the maximality of $|R|$, completing the proof.□

References

[1] N. Alon, *Combinatorial Nullstellensatz*, Combin. Probab. Comput. **8** (1999), 7–29.

[2] N. Alon, L. Babai and H. Suzuki, *Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems*, J. Combin. Theory Ser. A **58** (1991), 165–180.

[3] S. D. Adhikari, A. A. Ambily and B. Sury, *Zero-sum problems with subgroup weights*, preprint.

[4] S. D. Adhikari, R. Balasubramanian, F. Pappalardi & P. Rath, *Some zero-sum constants with weights*, Proc. Indian Acad. Sci. (Math. Sci.) **118**(2008), 183–188.

[5] S. D. Adhikari and Y. G. Chen, *Davenport constant with weights and some related questions II.*, J. Combin. Theory Ser. A **115** (2008), 178–184.

[6] S. D. Adhikari, Y. G. Chen, J. B. Friedlander, S. V. Konyagin and F. Pappalardi, *Contributions to zero–sum problems*, Discrete Math. **306** (2006), 1–10.

[7] Sukumar Das Adhikari and Purusottam Rath, *Davenport constant with weights and some related questions*, Integers **6** (2006), #A30 (electronic).

[8] L. Babai, P. Frankl, S. Kutin and Daniel Štefankovič, *Set systems with restricted intersections modulo prime powers*, J. Combin. Theory Ser A **95** (2001), 39–73.

[9] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory.* Pure and Applied Mathematics (Boca Raton), 278. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[10] W. D. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. **24** (2006), 337–369.

[11] P. Erdős, A. Ginzburg and A. Ziv, *Theorem in the additive number theory*, Bull. Research Council Israel **10F** (1961), 41–43.

[12] P. Frankl and R. M. Wilson, *Intersection theorems with geometric consequences*, Combinatorica **1** (1981), 357–368.

[13] D. J. Grynkiewicz, *A weighted Erdős-Ginzburg-Ziv theorem*, Combinatorica **26** (2006), 445–453.

[14] D. J. Grynkiewicz, O. Ordaz and E. Marchan, *A weighted generalization of two conjectures of Gao*, preprint, 2009.

[15] H. Harborth, *Ein Extremalproblem für Gitterpunkte*, J. Reine Angew. Math. **262/263** (1973), 356–360.

[16] Melvyn B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, New York, 1996.

[17] J. E. Olson, *A combinatorial problem in finite abelian groups, I*, J. Number Theory **1**(1969), 8–10.

[18] J. E. Olson, *A combinatorial problem in finite abelian groups, II*, J. Number Theory **1**(1969), 195-199.

[19] D. K. Ray-Chaudhuri and R. M. Wilson, *On t-designs*, Osaka J. Math. **12** (1975), no. 3, 737–744.

[20] C. Reiher, *On Kemnitz's conjecture concerning lattice points in the plane*, Ramanujan J. **13** (2007), 333–337.

[21] Z. W. Sun, *Unification of zero-sum problems, subset sums and covers of* $\mathbb{Z}$, Electron. Res. Announc. Amer. Math. Soc. **9** (2003), 51–60.

[22] Z. W. Sun, *On sums of binomial coefficients and their applications*, Discrete Math. **308** (2008), 4231–4245.

[23] Z. W. Sun, *Zero-sum problems for abelian p-groups and covers of the integers by residue classes*, Israel J. Math. **170** (2009), 235–252.

[24] R. Thangadurai, *A variant of Davenport's constant*, Proc. Indian Acad. Sci. (Math. Sci.) **117** (2003), 147–158.

[25] Xingwu Xia and Zhigang Li, *Some Davenport constants with weights and Adhikari & Rath's conjecture*, Ars Combin. **88** (2008), 83–95.

[26] P. Yuan and X. Zeng, *Davenport constant with weights*, European Journal of Combinatorics **31** (2010), no. 3, 677–680.

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