ON THE STRUCTURE OF MINIMAL ZERO-SUM SEQUENCES WITH MAXIMAL CROSS NUMBER

ALFRED GEROLDINGER AND DAVID J. GRYNKIEWICZ

Abstract. Let $G$ be an additive finite abelian group, $S = g_1 \cdot \ldots \cdot g_l$ a sequence over $G$ and $k(S) = \text{ord}(g_1)^{-1} + \ldots + \text{ord}(g_l)^{-1}$ its cross number. Then the cross number $K(G)$ of $G$ is defined as the maximal cross number of all minimal zero-sum sequences over $G$. In the spirits of inverse additive number theory, we study the structure of those minimal zero-sum sequences $S$ over $G$ whose cross number equals $K(G)$. These questions are motivated by applications in the theory of non-unique factorizations.

1. Introduction

Let $G$ be an additively written finite abelian group, $G = C_{q_1} \oplus \ldots \oplus C_{q_s}$ its direct decomposition into cyclic groups of prime power order, exp($G$) its exponent, and set

$$k^*(G) = \sum_{i=1}^{s} \frac{q_i - 1}{q_i} \quad \text{and} \quad K^*(G) = \frac{1}{\exp(G)} + k^*(G).$$

Note $k^*(G) = 0$ and $K^*(G) = 1$ for $G$ trivial. For a sequence $S = g_1 \cdot \ldots \cdot g_l$ over $G$,

$$k(S) = \sum_{i=1}^{l} \frac{1}{\text{ord}(g_i)} \in \mathbb{Q}$$

is the cross number of $S$, and

$$K(G) = \max \left\{ k(S) \mid S \text{ is a minimal zero-sum sequence over } G \right\}$$

denotes the cross number of the group $G$. It was introduced by U. Krause in 1984 (see [17], [18]) and since that time was studied under various aspects (see [16, 6, 7, 9, 2, 10, 1, 3]). The cross number may be viewed as a special weighted version of the Davenport constant. Its relevance stems from the theory of non-unique factorizations (see [20, 21, 24] and [8, Chapter 6]).

We trivially have $K^*(G) \leq K(G)$, and equality holds in particular for $p$-groups (see [8, Proposition 5.1.18 and Theorem 5.5.9]). Recently, B. Girard ([14]) established a new upper bound for the cross number, and his results support the conjecture that we always have equality.

In the present paper, we study the inverse problem associated to the cross number, that is we study the structure of minimal zero-sum sequences $U$ over $G$ with $k(U) = K(G)$. These investigations are motivated by questions from factorization theory (see recent work on $\Delta^*(G)$ performed in [22, 25]), and they are part of inverse additive number theory (see [19] for general information, and [5, Section 7] for a recent survey on inverse zero-sum problems). This inverse question is simple for cyclic groups of prime power order (see [8, Theorem 5.1.10]). The case when $G$ is a direct sum of an elementary $p$-group and an elementary $q$-group is studied in [11]. More recent progress is again due to B. Girard [12, 13]. The main results of the present paper (formulated in Theorems 3.7 and 3.9) give information on the order of elements contained in a minimal zero-sum sequence $U$ with $k(U) = K(G)$. The results are sharp for $p$-groups and almost sharp in the general case (see the discussion following Theorem 3.9). Among other

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consequences, they give a structural characterization of the crucial equality $K(G) = K^*(G)$ (see Theorem 3.14).

Throughout this article, let $G$ be an additively written, finite abelian group.

2. Preliminaries

Our notation and terminology are consistent with [4] and [8]. We briefly gather some key notions and fix the notation concerning sequences over finite abelian groups. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a,b \in \mathbb{R}$, we set $[a,b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. For $n \in \mathbb{N}$ and $p \in \mathbb{P}$, let $C_n$ denote a cyclic group with $n$ elements, $nG = \{ng \mid g \in G\}$ and $\nu_p(n) \in \mathbb{N}_0$ the $p$-adic valuation of $n$ with $\nu_p(p) = 1$. Throughout, all abelian groups will be written additively.

An $s$-tuple $(e_1, \ldots, e_s)$ of elements of $G$ is said to be independent (or briefly, the elements $e_1, \ldots, e_s$ are said to be independent) if $e_i \neq 0$ for all $i \in [1,s]$ and, for every $s$-tuple $(m_1, \ldots, m_s) \in \mathbb{Z}^s$,\[ m_1 e_1 + \ldots + m_s e_s = 0 \quad \text{implies} \quad m_1 e_1 = \ldots = m_s e_s = 0. \]

An $s$-tuple $(e_1, \ldots, e_s)$ of elements of $G$ is called a basis if it is independent and $G = \langle e_1 \rangle \oplus \ldots \oplus \langle e_s \rangle$. For a prime $p \in \mathbb{P}$, we denote by $G_p = \{g \in G \mid \text{ord}(g) \text{ is a power of } p\}$ the $p$-primary component of $G$, and by $r_p(G)$, the $p$-rank of $G$ (which is the rank of $G_p$).

Let $\mathcal{F}(G)$ be the free monoid with basis $G$. The elements of $\mathcal{F}(G)$ are called sequences over $G$. We write sequences $S \in \mathcal{F}(G)$ in the form

$S = \prod_{g \in G} g^{\nu_g(S)}$, with $\nu_g(S) \in \mathbb{N}_0$ for all $g \in G$.

We call $\nu_g(S)$ the multiplicity of $g$ in $S$, and we say that $S$ contains $g$ if $\nu_g(S) > 0$. A sequence $S_1$ is called a subsequence of $S$ if $S_1 \mid S$ in $\mathcal{F}(G)$ (equivalently, $\nu_g(S_1) \leq \nu_g(S)$ for all $g \in G$). If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence $S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\nu_g(S)} \in \mathcal{F}(G)$, we call

$|S| = l = \sum_{g \in G} \nu_g(S) \in \mathbb{N}_0$ \quad the length of $S$;

$\text{supp}(S) = \{g \in G \mid \nu_g(S) > 0\} \subset G$ \quad the support of $S$;

$\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} \nu_g(S)g \in G$ \quad the sum of $S$.

The sequence $S$ is called

- a zero-sum sequence if $\sigma(S) = 0$,
- zero-sum free if there is no nontrivial zero-sum subsequence,
- a minimal zero-sum sequence if $1 \neq S$, $\sigma(S) = 0$, and every $S' | S$ with $1 \leq |S'| < |S|$ is zero-sum free.

We denote by $A(G) \subset \mathcal{F}(G)$ the set of all minimal zero-sum sequences over $G$. Every map of abelian groups $\varphi : G \to H$ extends to a homomorphism $\varphi : \mathcal{F}(G) \to \mathcal{F}(H)$ where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. We say that $\varphi$ is constant on $S$ if $\varphi(g_1) = \ldots = \varphi(g_l)$. If $\varphi$ is a homomorphism, then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \text{Ker}(\varphi)$.

Let

$K \subset \mathcal{F}(G)$

be the set of all minimal zero-sum sequences over $G$. Throughout this article, let $G$ be an additively written, finite abelian group.
• \( D(G) \) denote the smallest integer \( l \in \mathbb{N} \) such that every sequence \( S \) over \( G \) of length \( |S| \geq l \) has a zero-sum subsequence.
• \( \eta(G) \) denote the smallest integer \( l \in \mathbb{N} \) such that every sequence \( S \) over \( G \) of length \( |S| \geq l \) has a zero-sum subsequence \( T \) of length \( |T| \in [1, \exp(G)] \).

Then \( D(G) \) is called the Davenport constant of \( G \), and we have \( D(G) = \max\{|U| \mid U \in \mathcal{A}(G)\} \). We use 
\[ k(G) = \max\{k(U) \mid U \in \mathcal{F}(G) \text{ zero-sum free} \} \in \mathbb{Q} \]
to denote the little cross number of \( G \). We summarize the main properties of \( k(G) \), \( D(G) \) and \( \eta(G) \) which will be used in the manuscript without further citing. Suppose that
\[ G = C_{n_1} \oplus \ldots \oplus C_{n_r} \text{ with } 1 < n_1 \mid \ldots \mid n_r \text{ and set } d^*(G) = \sum_{i=1}^{r} (n_i - 1). \]

Then we have trivially,
\[ k^*(G) \leq k(G), \quad K^*(G) \leq K(G) \quad \text{and} \quad 1 + d^*(G) \leq D(G). \]
Equality holds for \( p \)-groups ([8, Theorem 5.5.9]). Moreover, if \( G = C_{n_1} \oplus C_{n_2} \) with \( 1 \leq n_1 \mid n_2 \), then (see [8, Chapter 5.7])
\[ 1 + d^*(G) = D(G) \quad \text{and} \quad \eta(G) = 2n_1 + n_2 - 2. \]
The result on \( \eta(G) \) is based on the determination of the Erdős-Ginzburg-Ziv constant \( s(G) \), and thus on Reihner’s solution ([23]) of the Kemnitz conjecture. For groups with rank \( r \geq 3 \), no such results are available (see [5, Section 7] for more information around that).

### 3. Structural results

We start with some lemmas which are elementary and combinatorial. The main results, Theorems 3.7, 3.9 and 3.14, are based on Equation (1) and use inductive techniques from zero-sum theory (cf. [8, Chapter 5.7]).

**Proposition 3.1.**

1. There exists some \( U \in \mathcal{A}(G) \) with \( k(U) = K(G) \) such that 
\[ \max\{v_p(\exp(g)) \mid g \in \supp(U)\} = v_p(\exp(G)) \]
for all \( p \in \mathbb{P} \).

2. Let \( U \in \mathcal{A}(G) \) with \( k(U) = K(G) \). Then \( \supp(U) = G \) if and only if 
\[ \max\{v_p(\exp(g)) \mid g \in \supp(U)\} = v_p(\exp(G)) \]
for all \( p \in \mathbb{P} \).

**Proof.** 1. This follows from [8, Proposition 5.1.12.2].

2. If \( \supp(U) = G \) and \( p \in \mathbb{P} \), then 
\[ v_p(\exp(G)) = v_p(\supp(U)) = \max\{v_p(\exp(g)) \mid g \in \supp(U)\}. \]
Conversely, set \( H = \langle \supp(U) \rangle \) and assume to the contrary that \( H \leq G \). Clearly, the hypothesis implies that \( \exp(H) = \exp(G) \). Since \( H \neq G \), there exists \( p \in \mathbb{P} \) such that \( G_p \neq H_p \). Let \( g \in \supp(U) \) with 
\[ v_p(\exp(g)) = v_p(\exp(H)) = v_p(\exp(G)), \]
and let \( g = h + g' \), where \( g' \in H_p, h \in H \) and \( p \nmid \exp(h) \). Then \( \ord(g') = p^{\ord(\exp(G))} \), and thus \( g' \) is a direct summand in \( G_p \) so that \( G_p = G_p' \oplus \langle g' \rangle \), for some \( G_p' < G_p \). Consequently, since \( H_p \leq G_p \) and \( g' \in H_p \), there must exist some \( e \in G_p' \setminus H_p \).

Let \( g'' = g - (p - 1)e \). Since \( g = h + g' \) with \( p \nmid \exp(h) \) and \( \ord(g') = p^{\ord(\exp(G))} \), since \( G_p = G_p' \oplus \langle g' \rangle \), and since \( e \in G_p' \), we see (by considering \( v_q(\ord(g'')) \)) for all \( q \in \mathbb{P} \), with separate cases for \( q = p \) and \( q \neq p \) that \( \ord(g'') = \ord(g) \). Thus, since \( ne \notin H = \langle \supp(U) \rangle \) for \( n \in [1, p - 1] \) (in view of \( e \notin H \)), we have 
\[ U' = g''e^{p-1}U^{-1}g^{-1} \in \mathcal{A}(G) \]
with \( k(U') > k(U) = K(G) \), a contradiction. \( \square \)
Proposition 3.2. The following statements are equivalent:

(a) \( K(G) > K(H) \) for all proper subgroups \( H \leq G \).

(b) For all \( U \in \mathcal{A}(G) \) with \( k(U) = K(G) \), we have \( \max\{v_p(\ord(g)) \mid \text{ } g \in \text{supp}(U)\} = v_p(\exp(G)) \) for all \( p \in \wp \).

Proof. First we show (a) implies (b). Assume to the contrary that (b) fails. Then Proposition 3.1.2 implies that there exists \( U \in \mathcal{A}(G) \) with \( k(U) = K(G) \) and \( H = \langle \text{supp}(U) \rangle \leq G \). Thus \( U \in \mathcal{A}(H) \) and

\[
K(G) = k(U) \leq K(H) \leq K(G),
\]

contradicting (a).

Next we show (b) implies (a). Let \( H \leq G \) be a subgroup with \( K(H) = K(G) \). Then there exists some \( U \in \mathcal{A}(H) \) with \( k(U) = K(H) = K(G) \). Hence (b) and Proposition 3.1.2 imply that \( G = \langle \text{supp}(U) \rangle \leq H \), contradicting that \( H \leq G \).

We need the following basic result (see [15, Lemma 4.5]).

Lemma 3.3. Let \( G = C_{m_1} \oplus \ldots \oplus C_{m_r} \), with \( 1 < m_1 | \ldots | m_r \), and let \( H \leq G \) be a subgroup, say \( H \cong C_{m'_1} \oplus \ldots \oplus C_{m'_r} \), with \( 1 \leq m'_1 | \ldots | m'_r \). If \( m'_t = m_t \) for some \( t \in [1, r] \), then there exists a subgroup \( K \leq H \) such that \( K \cong C_{m_t} \), and \( K \) is a direct summand in both \( H \) and \( G \).

Proposition 3.4.

1. For a proper subgroup \( H \leq G \), the following statements are equivalent:
   (a) \( K^*(H) = K^*(G) \).
   (b) \( G \) is a \( p \)-group and \( H \) has a direct summand \( A \) such that \( H = A \oplus B \) and \( G = A \oplus C \), with \( B \cong C_{p^l} \) and \( C \cong C_{p^m} \), where \( p \in \wp \), \( m, l \in \mathbb{N}_0 \), \( l < m \) and \( \exp(A) \mid p^l \).

2. The following statements are equivalent:
   (a) \( G \) has a proper subgroup \( H \leq G \) with \( K^*(H) = K^*(G) \).
   (b) \( G = A \oplus C \cong C_{p^n} \), where \( p \in \wp \), \( m \in \mathbb{N} \) and \( A \leq G \) is a subgroup with \( \exp(A) \mid p^{m-1} \).

Proof. 1. Let \( H \leq G \) be a proper subgroup.

We write \( G \) in the form \( G \cong \bigoplus_{i=1}^l \left( \bigoplus_{j=1}^{l_i} C_{p_{i,j}} \right) \), with \( l \in \mathbb{N} \), \( p_1, \ldots, p_t \) distinct primes, \( 0 \leq k_{i,1} \leq \ldots \leq k_{i,r} \) for all \( i \in [1, l] \) and \( k_{1,r}, \ldots, k_{l,r} \in \mathbb{N} \). Then \( H \) may be written in the form \( H \cong \bigoplus_{i=1}^l \left( \bigoplus_{j=1}^{l_i} C_{p_{i,j}} \right) \), with \( 0 \leq k'_{i,1} \leq \ldots \leq k'_{i,r} \) and \( k'_{i,j} \leq k_{i,j} \), for all \( i \in [1, l] \) and all \( j \in [1, r] \) (see [8, Appendix A]). This shows that \( k^*(H) < k^*(G) \), and thus \( K^*(H) = K^*(G) \) implies that \( \exp(H) < \exp(G) \). Therefore we get \( k'_{i,r} < k_{i,r} \) for some \( i \in [1, l] \), say w.l.o.g.

\[
k'_{1,r} < k_{1,r}.
\]

Assume to the contrary that \( G \) is not a \( p \)-group (so \( l \geq 2 \)). First we suppose that \( H \) is a \( p \)-group. Then \( H \) is nontrivial, \( \exp(H) \geq 2 \), \( G = G_p \oplus A \) for a direct summand \( A \), and \( A \) has a direct summand isomorphic to \( C_q \) for some prime power \( q \). Then

\[
K^*(G) - K^*(H) \geq K^*(G_p) + \frac{q-1}{q} + \frac{1}{\exp(G)} - K^*(H) - \frac{1}{\exp(H)}
\]

\[
\geq \frac{q-1}{q} + \frac{1}{\exp(G)} - \frac{1}{\exp(H)} > 0,
\]

since \( \exp(H) \geq 2 \).
a contradiction. Thus $H$ is not a $p$-group and $p_1^{k_{1,r}} \cdots p_l^{k_{l,r}} \geq 2p_1^{k_{1,r}}$. Thus we get

$$K^*(G) - K^*(H) = \sum_{i=1}^{r} \sum_{j=1}^{k_{i,j} - k_{i,r}'} \frac{p_i}{p_i^{k_{i,j}}} - 1 + \frac{1}{\exp(G)} - \frac{1}{\exp(H)}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{k_{i,j} - k_{i,r}'} \frac{p_i}{p_i^{k_{i,j}}} - 1 + \frac{1}{p_1^{k_{1,r}'} \cdots p_l^{k_{l,r}'}} - \frac{k_{i,r}'}{p_1^{k_{1,r}'} \cdots p_l^{k_{l,r}'}}$$

$$\geq \frac{p_1^{k_{1,r}'} - k_{1,r}'}{p_1^{k_{1,r}'} - 1} + \frac{1}{p_1^{k_{1,r}'} \cdots p_l^{k_{l,r}'}} - \frac{k_{i,r}'}{p_1^{k_{1,r}'} \cdots p_l^{k_{l,r}'}}$$

$$> \frac{p_1^{k_{1,r}'} - k_{1,r}'}{p_1^{k_{1,r}'} - 1} - \frac{1}{p_1^{k_{1,r}'} \cdots p_l^{k_{l,r}'}} \geq \frac{p_1^{k_{1,r}'} - k_{1,r}'}{p_1^{k_{1,r}'} - 1} - \frac{1}{2p_1^{k_{1,r}'}} \geq 0,$$

a contradiction.

Thus it follows that $G$ is a $p$-group, and hence $l = 1$, $\exp(H) = p_1^{k_{1,r}'}$ and $\exp(G) = p_1^{k_{1,r}'}$. Hence

$$\sum_{i=1}^{r-1} \frac{p_i^{k_{i,i} - 1}}{p_i^{k_{i,i} - 1}} + 1 = K^*(H) = K^*(G) = \sum_{i=1}^{r-1} \frac{p_i^{k_{i,i} - 1}}{p_i^{k_{i,i} - 1}} + 1,$$

whence $k_{1,i}' = k_{1,i}$ for all $i \in [1, r - 1]$. Thus $H = A \oplus B$ with $B \cong C_{p_1^{k_{1,r}'}}$, $A \cong \bigoplus_{i=1}^{r-1} C_{p_i^{k_{i,i}'}}$ and $k_{1,r}' < k_{1,r}$. The conclusion for $H$ trivial now follows, and so we assume $H$ is nontrivial. Now if $k_{1,r}' = 0$, then $G \cong H \oplus C_{k_{1,r}'}$, whence

$$K^*(G) = 1 + k^*(H) > \frac{1}{\exp(H)} + k^*(H) = K^*(H),$$

contradicting (a). Therefore $1 \leq k_{1,r}' < k_{1,r}$, and so $k_{1,r} \geq 2$. Thus, if $r = 1$, then the result is complete using $A$ trivial. For $r \geq 2$, applying Lemma 3.3 to $H \leq G$, we see that there exists $A_1 \leq H$ with $A_1 \cong C_{p_1^{k_{1,1}'}}$ and $A_1$ a direct summand in both $H$ and $G$. Moreover, we can choose the complimentary summands so that $H = A_1 \oplus H'$ and $G = A_1 \oplus G'$ with $H' \leq G'$, and now by iterating this application of Lemma 3.3 (next to $H' \leq G'$, and then so forth), we see that an appropriate isomorphic copy of $A$ can be chosen that is a direct summand in both $H$ and $G$, which establishes (b).

Next we show (b) implies (a). Since $\exp(H) = p^l$ and $\exp(G) = p^m$, the hypotheses of (b) further imply

$$K^*(H) = k^*(A) + 1 = K^*(G),$$

desired.

2. This follows immediately from 1 (to see (b) implies (a), take $H = A \oplus p^{m-1}C \cong A \oplus C_{p^l}$, where $l = v_p(\exp(A))$).

For the proof of Theorem 3.9 we will need the following two lemmas. A sequence $S \in F(G)$ is called short (in $G$) if $1 \leq |S| \leq \exp(G)$.

**Lemma 3.5.** Let $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 | n_2$ and let $S \in F(G)$ be a zero-sum sequence of length $|S| > D(G) = n_1 + n_2 - 1$. Then $S$ has a short zero-sum subsequence.

**Proof.** Since $\eta(G) = 2n_1 + n_2 - 2$ and $D(G) = n_1 + n_2 - 1$ (see Equation (1)), the assertion is clear for $|S| \geq 2n_1 + n_2 - 2$. Suppose that $|S| = n_1 + n_2 + k$ with $k \in [0, n_1 - 1]$. By [4, Theorem 6.7],

$$S_0 = 0^{n_2 - k - 2} \cdot k$$
has a zero-sum subsequence \( W = 0^l S' \) of length in \( \{n_2, 2n_2\} \), where \( l \in [0, n_2 - 2 - k] \) and \( S' \in \mathcal{F}(G) \) is a zero-sum subsequence of \( S \). If \( |W| = n_2 \), then \( S' \) is a short zero-sum subsequence of \( S \). If \( |W| = 2n_2 \),

then \( S'' \) is a zero-sum subsequence of \( S \) of length

\[
|S| - |S'| = n_1 + n_2 + k - (2n_2 - l) \leq n_1 + n_2 + k - 2n_2 + n_2 - 2 - k = n_1 - 2 \leq n_2.
\]

□

**Lemma 3.6.** If \( H = \mathbb{H} \oplus K \), with \( H \) and \( K \) nontrivial, then \( K(G) \geq K(H) \).

**Proof.** Let \( U \in \mathcal{A}(H) \) with \( k(U) = K(H) \) and \( |U| > 1 \) (possible in view of \( H \) nontrivial and Proposition 3.1.1), let \( h \in \text{supp}(U) \), let \( g \in K \setminus \{0\} \), and let \( h' = h - (\text{ord}(g) - 1)g \). Set

\[
U' = g^{\text{ord}(g)-1}h'h^{-1}.
\]

Then \( U' \in \mathcal{A}(G) \) with

\[
k(U') \geq k(U) - \frac{1}{\text{ord}(h)} + \frac{\text{ord}(g) - 1}{\text{ord}(g)} + \frac{1}{\text{ord}(h')} \geq k(U) - \frac{1}{2} + \frac{1}{2} + \frac{1}{\text{ord}(h')} > k(U) = K(H),
\]

whence \( K(G) > K(H) \) follows. □

**Theorem 3.7.** Let \( G = C_{q_1} \oplus \cdots \oplus C_{q_s} \), where \( 1 < q_1 \leq \cdots \leq q_s = p^m \) are prime powers with \( s, m \in \mathbb{N} \) and \( p \in \mathbb{P} \). Suppose that

\[
G_p = C_{p^m_1} \oplus \cdots \oplus C_{p^m_r}, \quad \text{where} \quad 1 \leq m_1 \leq \cdots \leq m_r = m,
\]

and, for every \( i \in [1, s] \), let \( H_i < G \) be such that \( G = H_i \oplus C_{q_i} \). Let \( U \in \mathcal{A}(G) \) with \( k(U) = K(G) \) and \( |U| = l \). For every \( i \in [1, s] \), we set

\[
U = (h_{i,1} + a_{i,1}) \cdots (h_{i,l} + a_{i,l}) \quad \text{where} \quad h_{i,\nu} \in H_i \quad \text{and} \quad a_{i,\nu} \in C_{q_i} \quad \text{for all} \ \nu \in [1, l],
\]

and let

\[
\alpha_i = \max\{\text{ord}(a_{i,\nu}) \mid \nu \in [1, l]\}.
\]

1. For all \( i \in [1, s - 1] \), we have \( \alpha_i = q_i \), and if \( s \geq 2 \), then \( \alpha_s \geq q_{s-1} \).

2. If \( G \neq G_p \) and \( \alpha_s < q_s \), then \( r \geq 3 \) and \( 2 \leq m_r - 2 \leq m_{r-1} < m_r \).

**Definition 3.8.**

1. We say that \( G \) is exceptional if it has the form given in Proposition 3.4.2.

2. A sequence \( U \) satisfying the hypotheses of Theorem 3.7.2 will be called anomalous over \( G \).

**Theorem 3.9.** Let all notation be as in Theorem 3.7, and suppose \( G \neq G_p \). Let \( q \) be a prime divisor of \( \exp(G) \), \( V \) the subsequence of \( U \) consisting of all terms \( g \) with \( v_q(\text{ord}(g)) = \gamma \) maximal, and suppose that \( (\text{supp}(V)) = K \) is a \( q \)-group. Then \( |V| \geq D(q^{-1}K) \), and if \( t_q(gG) \leq 2 \), then even

\[
|V| \geq D(q^{-1}K) + q.
\]

Let all notations be as above. If \( G \) is a \( p \)-group, then \( \alpha_s \geq p^{m_r - 1} \) cannot be strengthened in general. This will be shown in Corollary 3.12. Suppose that \( G \) is not a \( p \)-group. In that case we conjecture that \( \alpha_s = p^{m_r} \), in other words, there are no anomalous sequences (in order to show this, it suffices to consider the case \( \alpha_s = p^{m_r - 1} \), as will be seen in the following proof). The fact that the conjecture holds for \( r \leq 2 \) is heavily based on Equation (1), and note that no similar results are available for \( r \geq 3 \).

**Proof of Theorems 3.7 and 3.9.** We set \( \exp(G) = n \) and first proceed with a series of assertions.

**A1.** Let \( q \) be a prime divisor of \( n \) and \( \alpha = \max\{v_q(\text{ord}(g)) \mid g \in \text{supp}(U)\} \). If there exists some \( g \in \text{supp}(U) \) with \( \text{ord}(g) > \sum_{\nu=\beta}^\alpha q^\nu = \frac{q^{\alpha+1}-q^\beta}{q-1} \), where \( \beta = v_q(\text{ord}(g)) \), then \( \alpha = v_q(n) \).
A2. There exists at most one prime divisor \( q \) of \( n \) such that \( \max \{ v_q(\text{ord}(g)) \mid g \in \text{supp}(U) \} < v_q(n) \).

A3. Let \( q \) be a prime divisor of \( n \) such that \( \max \{ v_q(\text{ord}(g)) \mid g \in \text{supp}(U) \} = v_q(n) \). Suppose that \( G = H \oplus C_{q^\delta} \) and \( U = (h_1 + a_1) \cdots (h_l + a_l) \), where \( h_i \in H \) and \( a_i \in C_{q^\delta} \) for all \( i \in [1,l] \). Then \( \max \{ \text{ord}(a_i) \mid i \in [1,l] \} = q^\delta \).

A4. Let \( q' \) and \( q \) be two prime divisors of \( n \) with \( q'^{\alpha_0(n)} > q^{\alpha_0(n)} \). Then \( \max \{ v_{q'}(\text{ord}(g)) \mid g \in \text{supp}(U) \} = v_{q'}(n) \).

Proof of A1. Assume to the contrary that \( \alpha < v_q(n) \). Hence there is some \( e \in G \) with \( \text{ord}(e) = q^{\alpha_0+1} \). We set \( \text{ord}(g) = q^{\alpha t} \) for some \( t \in \mathbb{N} \) and \( g' = g - (q-1)e \). Then \( \text{ord}(g') = q^{\alpha_0+1} t \) and

\[ U' = g' e^{\alpha-1} U g^{-1} \in A(G) \]

with

\[ k(U') = k(U) - \frac{1}{q^{\alpha t}} + \frac{q-1}{q^{\alpha+1}} + \frac{1}{q^{\alpha_0+1} t} > k(U) = K(G), \]

a contradiction. \( \square \)

Proof of A2. Assume to the contrary that there are two distinct primes \( q' \) and \( q \) such that

\[ \alpha = \max \{ v_{q'}(\text{ord}(g)) \mid g \in G \} < v_q(n) \quad \text{and} \quad \beta = \max \{ v_q(\text{ord}(g)) \mid g \in G \} < v_q(n). \]

Without restriction we may suppose that \( q'^{\alpha+1} > q^{\beta+1} \). Hence \( \alpha < v_q(n) \), A1 implies that there exists some \( g \in \text{supp}(U) \) with \( \text{ord}(g) = q^\alpha \) (any \( g \in \text{supp}(U) \) with \( v_{q'}(g) = \alpha \) will do).

Choose elements \( c_1 \in G \) with \( \text{ord}(c_1) = q'^{\alpha_0+1} \), and \( c_2 \in G \) with \( \text{ord}(c_2) = q^{\alpha_0+1} \). Then we have \( \text{ord}(g - (q'-1)c_1 - (q-1)c_2) = q^{\alpha_0+1} q^{\beta+1} \) and

\[ U' = (g - (q'-1)c_1 - (q-1)c_2) e_1^{q'-1} e_2^{q-1} U g^{-1} \in A(G) \]

with

\[ k(U') = k(U) - \frac{1}{q^\alpha} + \frac{q'-1}{q^{\alpha+1}} + \frac{q-1}{q^{\beta+1}} + \frac{1}{q^{\alpha_0+1} q^{\beta+1}} > k(U) = K(G), \]

a contradiction. \( \square \)

Proof of A3. Assume to the contrary that \( \max \{ \text{ord}(a_i) \mid i \in [1,l] \} = q^\alpha < q^\delta \). Since \( \max \{ v_q(\text{ord}(g)) \mid g \in \text{supp}(U) \} = v_q(n) \), let \( g \in \text{supp}(U) \) with \( v_q(\text{ord}(g)) = v_q(n) \geq \delta > \alpha \). We pick some \( a_0 \in C_{q^\delta} \) with \( \text{ord}(a_0) = q^{\alpha+1} \) and set \( g' = g - (q-1)a_0 \). Then \( \text{ord}(g') = \text{ord}(g) \) and

\[ U' = a_0^{q-1} g' U g^{-1} \in A(G) \]

with \( k(U') > k(U) = K(G) \), a contradiction. \( \square \)

Proof of A4. Assume to the contrary that \( \max \{ v_{q'}(\text{ord}(g)) \mid g \in \text{supp}(U) \} = \alpha < v_{q'}(n) \). Then by A2, there exists some \( g \in \text{supp}(U) \) with

\[ \text{ord}(g) \geq q'^{\alpha_0(n)} > q^{\alpha_0(n)} \geq q^{\alpha+1} > \sum_{\nu=0}^{\alpha} q^{\nu}. \]

and hence A1 gives a contradiction. \( \square \)

In view of A4 and A3, we see that \( \alpha_i = q_i \), for all \( q_i \), such that \( p \mid q_i \). Set

\[ \alpha = \max \{ v_p(\text{ord}(g)) \mid g \in \text{supp}(U) \}. \]

We now proceed to show Theorem 3.7.1 holds. To that end, we can assume \( \alpha < v_p(n) \), else A3 implies 1. Thus A1 implies that all elements \( g \in \text{supp}(U) \) with \( v_p(\text{ord}(g)) = \alpha \) have \( \text{ord}(g) = p^\alpha \).

We continue with the following assertion, which establishes the first part of Theorem 3.7.1.

A5. If \( p \mid q_i \) and \( i \in [1,s-1] \), then \( \alpha_i = q_i \).
Proof of A5. Assume to the contrary that $\alpha_i = p^\beta < p^{\nu(q)}$. We pick some $a_0 \in C_{p^{\nu(q)}}$ with $\text{ord}(a_0) = p^{\beta+1}$ and some $e \in C_{p^n} = C_q$, with $\text{ord}(e) = p^n$ and set $g' = g - (p-1)a_0 - (p^{m\alpha} - 1)e$, where $g \in \text{supp}(U)$ with $\text{ord}(g) = p^\alpha$ (possible in view of the comment before the statement of A5). Then $\text{ord}(g') = p^m$ and
\[ a_0^{-1}p^{m-\alpha-1}Ug^{-1} \]
is zero-sum free. Thus
\[ U' = g'a_0^{-1}p^{m-\alpha-1}Ug^{-1} \in A(G) \]
and
\[ k(U') = k(U) - \frac{1}{p^\alpha} + \frac{p^{m-\alpha} - 1}{p^m} + \frac{1}{p^m} + \frac{p-1}{p^{\beta+1}} = k(U) + \frac{p-1}{p^{\beta+1}} > k(U) = K(G), \]
a contradiction. □

Now suppose $r = 1$. Then we may further assume $G \neq G_p$, else the proof of Theorem 3.7.1 is complete. Let $S$ denote the subsequence of $U$ consisting of all elements of order $p^\alpha$. Since $\sigma(U) = 0$, it follows that $\text{ord}(\sigma(S)) \leq p^{\alpha-1}$. Since $G \neq G_p$, there is a prime divisor $q$ of $n$ distinct from $p$, and in view of A3 and A4, there is some $h \in \text{supp}(U)$ with $\nu_q(h) = \nu_q(n)$. Consequently, $S$ is a proper subsequence of $U$. Thus, since $r = 1$, it follows that $S$ has a subsequence $T$ of length $|T| \leq D(p^{m-1}C_{p^n}) = p^s$ such that $\text{ord}(\sigma(T)) \leq p^{r-1}$ and which is a proper subsequence of $U$. We consider the sequence
\[ U' = T^{-1}\sigma(T)U \in A(G). \]
Then clearly $k(U') \geq k(U)$. Iterating this process, we either eventually obtain a sequence $W \in A(G)$ with $k(W) \geq k(U) = K(G)$ which satisfies the assumptions of A1, or else we find a proper zero-sum subsequence. In the second case, we contradict that $U \in A(G)$, and in the first, A1 implies that $\alpha = \nu_q(n)$, a contradiction. This completes the proof of 1. Note these arguments also show that $\alpha_s = q_s$ when $r = 1$ and $G \neq G_p$, handling one of the cases needed later for Theorem 3.7.2.

So we may now assume $r \geq 2$. To conclude the proof of Theorem 3.7.1, suppose to the contrary that $\alpha_s < p^{m\alpha-1}$ and $r \geq 2$. Then $\alpha = m\gamma_{r-1}$ (in view of A5). Let $g \in \text{supp}(U)$ with $\nu_q(\text{ord}(g)) = m\gamma_{r-1} = \alpha$ and pick some $a_0 \in C_q$, with $\text{ord}(a_0) = p^\alpha$. We set $g' = g - (p-1)a_0$. Then, since $\alpha_s < p^\alpha = p^{m\alpha-1}$, $g \in \text{supp}(U)$ and $a_0 \in C_q$, it follows that $\text{ord}(g') = \text{ord}(g)$ and

\[ U' = a_0^{-1}g'Ug^{-1} \in A(G), \]

with $k(U') > k(U) = K(G)$, a contradiction. Thus Theorem 3.7.1 is established.

Next we proceed with the proof of Theorem 3.9. Thus, let $q$ be a prime divisor of $\exp(G)$, $V$ the subsequence of $U$ consisting of all terms $g$ with $\nu_q(\text{ord}(g)) = \gamma$ maximal, and suppose that $|\text{supp}(V)| = K$ is a q-group so that $q^{-1}K$ is an elementary q-group. Thus we have $\text{ord}(g) = q^\gamma$ for all $g \in \text{supp}(V)$. Since $\sigma(U) = 0$, it follows from the definition of $V$ that $\nu_q(\text{ord}(\sigma(V))) < \gamma$. Let $\varphi : G \to G$ be the multiplication by $q^{-1}$ map. Then $\varphi(K) = C_{q^\gamma}$, where $\theta$ is the rank of $q^{-1}K$, and $D(q^{-1}K) = \theta(q - 1) + 1$.

Let $e_1, \ldots, e_\theta \in G$ be independent elements such that $(\varphi(e_1), \ldots, \varphi(e_\theta))$ is a basis of $\varphi(K)$ (and thus $\text{ord}(e_i) = p^\gamma$ for all $i \in [1, \theta]$). Since $\nu_q(\text{ord}(\varphi(V))) < \gamma$ and $K$ is a q-group, it follows that we can factor $V = V_1V_2 \cdots V_\theta$ with $w \in \mathbb{N}$ and $\nu_q(V_i) \in A(\varphi(K))$ for all $i \in [1, \theta]$.

Let $U'' \in \mathcal{F}(G)$ be the sequence obtained from $U$ by replacing each subsequence $V_i$ by the single term $\sigma(V_i)$, i.e., $U'' = \sigma(V_1) \cdots \sigma(V_\theta)UV^{-1}$, let
\[ \beta = \max \{\nu_q(\text{ord}(g)) \mid g \in \text{supp}(U'')\}, \]
and let $g \in \text{supp}(U'')$ be an element such that $\text{ord}(g) = tq^\beta$ with $t$ maximal. Note $U'' \in A(G)$ (since $U \in A(G)$) and $\beta < \gamma$ (since $\nu_q(\text{ord}(\sigma(V))) < \gamma$). Let $g' = g - (q-1)\sum_{i=1}^\theta e_i$. Observe that $\text{ord}(g') = tq^\gamma$. Define
\[ U''' = e_1^{q^{-1}} \cdots e_\theta^{q^{-1}} g'U'g^{-1}. \]
Since $v_q(e_i) = \gamma > \beta$, it follows that $U'' \in \mathcal{A}(G)$. Moreover, since $\text{ord}(\sigma(V_i))|q^3$ for all $i \in [1, w]$, it follows that

$$k(U) = K(G) \geq k(U'') \geq k(U) - \frac{|V|}{q^3} - \frac{1}{tq^3} + \frac{w}{q^3} + \frac{1}{tq^3},$$

which implies that

$$|V| \geq \theta(q - 1) + \frac{1}{t} + q^{\gamma - \beta}(w - \frac{1}{t}).$$

Likewise, since $U' \in \mathcal{A}(G)$, we have

$$k(U) = K(G) \geq k(U') \geq k(U) - \frac{|V|}{q^3} + \frac{w}{q^3},$$

which implies

$$|V| \geq q^{\gamma - 3}w.$$

Suppose $w \geq 2$. Then $\gamma > \beta$ and $t \geq 1$ combined with (3) imply

$$|V| \geq \theta(q - 1) + 1 + q(2 - 1) = \theta(q - 1) + q + 1,$$

yielding (2) and so completing the proof of Theorem 3.9. So we may instead assume $w = 1$. Consequently (from the definitions of $w$ and $D(G)$), it follows that

$$|V| \leq D(\varphi(K)) = \theta(q - 1) + 1.$$

Suppose $\theta = 1$. Then we must have equality in (5), and thus in (4) as well, with $\beta = \gamma - 1$, else (6) is contradicted. However, equality in (4) implies that $U'$ is anomalous over $G$, whence Theorem 3.7.1 implies $q = p$. However, since Theorem 3.7.2 implies that there are no anomalous sequence over $G$ with $r_p(qG) \leq 2$, we see that this case will be complete once we have proved Theorem 3.7.2 (whose proof will only use the case $\theta \geq 2$ in the case 3.9). So we may assume $\theta \geq 2$.

Suppose $\theta \geq 3$. Let $W$ be the subsequence of $U$ consisting of all terms $h$ with $v_q(\text{ord}(h)) > 0$. Then, since $r_q(qG) \leq 2$, it follows that $\gamma = 1$, and thus all $h \in \text{supp}(W)$ have $\text{ord}(h)|q^1$. As a result, since $\sigma(U) = 0$, it follows that $\sigma(W) = 0$. Thus, since $U \in \mathcal{A}(G)$, it follows that either $W$ is trivial or $W = U$. Since $G \neq G_q$, and $k(U) = K(G)$, either case contradicts Lemma 3.6. So we may assume $\theta = 2$.

Let $f_1^{(0)}, f_2^{(0)} \in \text{supp}(V)$ be a basis for $K$. Let $\beta' \in [1, \gamma - 1]$ be the largest integer such that there is some $g \in \text{supp}(U)$ with $v_q(\text{ord}(g)) = \beta'$ and $\text{ord}(g) > q^2$; note, since $G \neq G_q$, that $\beta'$ must exist, else we obtain from Lemma 3.6 a contradiction to $k(U) = K(G)$, just as we did in the case $\theta \geq 3$. Furthermore, since $r_q(qG) \leq 2$, it follows that there are no three independent elements of order $q^2$ with $\beta'$. We iterate the arguments used to construct $U'$ and $U''$. Let $S_0 = V, U_0 = U$ and $\gamma_0 = \gamma$. Assuming $S_{j-1}, U_{j-1}, \gamma_{j-1} > \beta', f_1^{(j-1)}$ and $f_2^{(j-1)}$ have already been constructed, for $j \geq 1$, we define $S_j, U_j, \gamma_j, f_1^{(j)}$ and $f_2^{(j)}$ as follows. Since $v_q(\text{ord}(\sigma(S_{j-1}))) < \gamma_{j-1}$ and $v_q(\text{ord}(h)) \leq \gamma_{j-1}$ for all $h \in \text{supp}(S_{j-1})$ (this holds for $j - 1 = 0$ and follows, for $j - 1 \geq 1$, from the subsequent definitions of $S_j$ and $\gamma_j$), it follows from Lemma 3.5 (applied to $S_{j-1}$ modulo the multiplication by $p^{\gamma_j - 1 - 1}$ homomorphism; we are allowed to apply it in view of $\gamma_{j-1} > \beta'$ and the conclusion of the previous paragraph) that we can factor $S_{j-1} = V_1^{(j-1)} \cdot \ldots \cdot V_{w_{j-1}}^{(j-1)}$ with $\sigma(V_i) \in q^{v_q(\text{ord}(h)) - \gamma_{j-1} + 1}G_q$ for all $i$ and with $1 \leq |V_i| \leq q$ for $i \geq 2$.

Let $U_j = \sigma(V_1^{(j-1)}) \cdot \ldots \cdot \sigma(V_{w_{j-1}}^{(j-1)})U_{j-1}S_{j-1}^{-1}$, let $\gamma_j = \max\{v_q(\text{ord}(g)) | g \in \text{supp}(U_j)\}$, let $S_j$ be the subsequence of $U_j$ consisting of all terms $h$ with $v_q(\text{ord}(h)) = \gamma_j$, and let $f_1^{(j)}$ and $f_2^{(j)}$ be two independent elements of order $q^{\gamma_j}$. If $\gamma_j = \beta'$, stop. Otherwise, every element $h \in \text{supp}(U_j)$ with $v_q(\text{ord}(h)) = \gamma_j$ has $\text{ord}(h) = q^{\gamma_j}$, whence $\sigma(U_j) = \ldots \ldots$
\( \sigma(U_{j-1}) = \ldots = \sigma(U_0) = \sigma(U) = 0 \) implies \( v_q(\text{ord}(\sigma(S_j))) < \gamma_j \), as claimed previously. Let \( k \) be the index such that \( \gamma_k = \beta' \) (the process must terminate as \( \gamma_j \) decreases with each iteration and \( v_q(n) \) is finite).

By their construction, we have \( U_j \in \mathcal{A}(G) \) for all \( j \). Let \( g \in \text{supp}(U_k) \) with \( \text{ord}(g) = tq^\gamma \) and \( t \geq 2 \) (possible in view of the definition of \( \beta' \)). Then define

\[
U'' = f \cdot \prod_{i=0}^{k-1} (f_1^{(i)} f_2^{(i)})^{\theta-1} \cdot U_k g^{-1},
\]

where \( f = g - (q-1) \sum_{i=0}^{k-1} (f_1^{(i)} + f_2^{(i)}) \). Since \( \beta' = \gamma_k < \gamma_{k-1} < \ldots < \gamma_1 < \gamma_0 = \gamma \), since \( v_q(\text{ord}(h)) \leq \gamma_k \) for all \( h \in \text{supp}(U_k) \), and since \( U_k \in \mathcal{A}(G) \), it follows that \( U''' \in \mathcal{A}(G) \). Observe that \( \text{ord}(f) = tq^{\gamma_0} \). Thus, since \( |V_i'| \leq q \) for \( i \in [2, w_j] \) and \( |V_i'| \leq D(C_q \oplus C_q) = 2q - 1 \), for all \( j \in [0, k-1] \), since \( \gamma_i - 1 \geq \gamma_{i+1} \), for \( i \in [0, k-1] \), since \( t \geq 2 \), and since \( k \geq 1 \), it follows that

\[
K(G) \geq k(U''') \geq k(U) + \sum_{i=0}^{k-1} \left(-\frac{|S_i|}{q^{\gamma_i}} + \frac{w_i}{q^{\gamma_{i+1}}} \right) - \frac{1}{tq^{\gamma_k}} + \sum_{i=0}^{k-1} \left(2q - 2\right) \frac{1}{q^{\gamma_i}} + \frac{1}{tq^{\gamma_0}}
\]

\[
\geq k(U) + \sum_{i=0}^{k-1} \left(-\frac{2q - 1}{q^{\gamma_i}} + \frac{w_i}{q^{\gamma_{i+1}}} \right) - \frac{1}{tq^{\gamma_k}} + \sum_{i=0}^{k-1} \left(2q - 2\right) \frac{1}{q^{\gamma_i}} + \frac{1}{tq^{\gamma_0}}
\]

\[
= k(U) + \sum_{i=0}^{k-1} \left(-\frac{1}{q^{\gamma_i}} + \frac{1}{q^{\gamma_{i+1}}} \right) - \frac{1}{tq^{\gamma_k}} + \sum_{i=0}^{k-1} \left(2q - 2\right) \frac{1}{q^{\gamma_i}} + \frac{1}{tq^{\gamma_0}}
\]

\[
= k(U) - \frac{1}{q^{\gamma_0}} + \frac{1}{q^{\gamma_k}} + \frac{1}{tq^{\gamma_k}} + \frac{1}{tq^{\gamma_0}} > k(U) = K(G),
\]

a contradiction. Thus it remains to prove Theorem 3.7.2.

To this end, assume \( G \neq G_p \) and \( \alpha_s < q_s \). We can assume \( \alpha < v_p(n) \), else \textbf{A3} contradicts the hypotheses of Theorem 3.7.2. Thus \textbf{A1} implies that all elements \( g \in \text{supp}(U) \) with \( v_p(\text{ord}(g)) = \alpha \) have \( \text{ord}(g) = p^n \). In view of the previous work for \( r = 1 \), we may assume \( r \geq 2 \) as well. Furthermore, applying the argument used in the case \( r = 1 \), we may w.l.o.g. assume \( \alpha_s = p^{m_{s-1}} \). Thus \( \alpha = m_{s-1} \). Let \( V \) be as defined in the hypothesis of Theorem 3.9 with \( q = p \), and let \( \theta \) be as in the proof of Theorem 3.9. Note, in view of \( r \geq 2 \), \( G \neq G_p \) and Theorem 3.7.1, that \( V \) is a nontrivial, proper subsequence of \( U \).

Let \( \theta' = r_p(p^{m_{s-1}} G) \). Note \( \theta \leq \theta' \) and \( q_i \leq p^{m_{s-1}} \) for at least \( \theta' - 1 \) indices \( i \in [1, s] \). Suppose \( \theta < \theta' \). Then there exist \( g_1, \ldots, g_s \in \text{supp}(U) \) such that all elements \( h \in \text{supp}(U) \) with \( \text{ord}(h) = m_{s-1} \), we may apply the completed case of Theorem 3.9 to \( G \) with \( p = q \) to
conclude $|V| \geq 3p - 1$. If there are three independent elements of order $p^{m_{r-1}}$, then $r_p(pG) \leq 2$ implies $m_{r-1} = 1$, whence (in view of every $h \in \text{supp}(V)$ having $\text{ord}(h)|p^{m_{r-1}}$ and $\sigma(U) = 0$) $V$ is a zero-sum subsequence, which contradicts that $U \in \mathcal{A}(G)$ (we noted in a previous paragraph that $V$ is proper and nontrivial). Therefore we may assume there are no three independent elements of order $p^{m_{r-1}}$. Consequently, $|V| \geq 3p - 1 > \eta(C_p^2)$ implies that we can find a subsequence $V_0|V$ with $|V_0| \leq p$ and $\text{ord}(\sigma(V_0))|p^{m_{r-1}-1}$. Therefore the sequence

$$U' = \sigma(V_0)UV_0^{-1} \in \mathcal{A}(G)$$

satisfies

$$K(G) \geq k(U') \geq k(U) - \frac{|V_0|}{p^{m_{r-1}}} + \frac{1}{p^{m_{r-1}-1}} \geq k(U) = K(G)$$

and hence $k(U') = K(G)$. Iterating this process, we see that we can w.l.o.g. assume

$$p < 2p - 1 = (3p - 1) - p \leq |V| < 3p - 1,$$

which contradicts Theorem 3.9 applied to $V$ one last time, completing the proof. \hfill $\Box$

**Corollary 3.10.** Suppose that $G$ is not a $p$-group and let $U \in \mathcal{A}(G)$ with $k(U) = K(G)$. Then for every prime divisor $p$ of $\exp(G)$ with $r_p(pG) \leq 2$, there exists some $g \in \text{supp}(U)$ with $\text{ord}(g) = p^{s_t(\exp(G))}t$ for some $t \geq 2$.

**Proof.** Let $V$ be the subsequence of $U$ consisting of all $g \in \text{supp}(U)$ with $v_p(\text{ord}(g)) = v_p(\exp(G))$. By Theorem 3.7 and $r_p(pG) \leq 2$, we conclude that $V$ is nontrivial and proper (since $G \neq C_p$). Thus, if the corollary is false, then we can apply Theorem 3.9 to $U$ to conclude that

$$|V| \geq \theta(p - 1) + p + 1,$$

where $\theta = r_p(K)$ and $K = \langle \text{supp}(V) \rangle$. If $\theta \geq 3$, then $r_p(pG) \leq 2$ implies that $v_p(\exp(G)) = 1$, whence $V$ is a zero-sum subsequence of $U$, contradicting that $U \in \mathcal{A}(G)$. Therefore $\theta \leq 2$, and we see that $|V| > \eta(C_p^2)$ (recall $\eta(C_p^2) = 3p - 2$ and $\eta(C_p) = p$ by (1)). Thus we can find $V_0|V$ such that $\text{ord}(\sigma(V_0))|p^{s_t(\exp(G))}t - 1$. Defining $U' = \sigma(V_0)UV_0^{-1} \in \mathcal{A}(G)$, observe, as in the proof of Theorem 3.7.2, that $k(U') = k(U) = k(G)$. Thus iterating this process, we can reduce the length of $V$ until $|V| < \eta(C_p^0)$, which then contradicts Theorem 3.9 applied once more, completing the proof. \hfill $\Box$

The following corollary is thought to likely hold for all $G$. Here we show a very special case.

**Corollary 3.11.** Suppose $\exp(G) = p^\alpha q$ and $r_p(pG) \leq 2$, where $p, q \in \mathbb{P}$ and $\alpha \geq 0$. Then

$$K(G) = \frac{1}{\exp(G)} + k(G).$$

**Proof.** By the results mentioned at the end of Section 2, the result holds for $p$-groups. Therefore we may suppose that $p$ and $q$ are distinct and that $\alpha \geq 1$. Let $U \in \mathcal{A}(G)$ with $k(U) = K(G)$. Applying Corollary 3.10 to $U$, we find that there is some $g \in \text{supp}(U)$ with $\text{ord}(g) = \exp(G)$. Thus the assertion follows from [8, Proposition 5.1.8.6]. \hfill $\Box$

**Corollary 3.12.** Let $G = C_{p^{m_1}} \oplus \ldots \oplus C_{p^{m_r}}$ be a $p$-group with $p \in \mathbb{P}, r \in \mathbb{N}$ and $1 \leq m_1 \leq \ldots \leq m_r$.

1. For every $m \in [m_{r-1}, m_r]$, there exists some $U \in \mathcal{A}(G)$ with $k(U) = K(G)$ and $\max\{\text{ord}(g) \mid g \in \text{supp}(U)\} = p^m$.

2. $G$ is not exceptional if and only if every $U \in \mathcal{A}(G)$ with $k(U) = K(G)$ contains some $g \in G$ with $\text{ord}(g) = \exp(G)$.
Proof. 1. Let $m \in [m_r-1, m_r]$ and let $(e_1, \ldots, e_r)$ be a basis of $G$ with $\text{ord}(e_i) = p^{m_i}$ for $i \in [1, r]$. We set $e'_r = p^{m_r-m}e_r$ and $e_0 = e_1 + \ldots + e_r + e'_r$. Then $\text{ord}(e'_r) = \text{ord}(e_0) = p^m$ and

$$U = e_0 e'_r p^{m-1} \prod_{v=1}^{r-1} e'_v p^{m_v-1} \in \mathcal{A}(G)$$

with $k(U) = 1 + \sum_{i=1}^{r-1} \frac{p^{m_i-1}}{p^{m_i}} = K^*(G) = K(G)$.

2. By definition, $G$ is not exceptional if and only if $r \geq 2$ and $m_r = m_r$. In that case, Theorem 3.7 implies that every $U \in \mathcal{A}(G)$ with $k(U) = K(G)$ contains some element $g \in G$ with $\text{ord}(g) = \exp(G)$. Conversely, if $G$ is exceptional, then Corollary 3.12.1 shows, for $r \geq 2$, that there exists some $U \in \mathcal{A}(G)$ with $k(U) = K(G)$ and $\max\{\text{ord}(g) \mid g \in \text{supp}(U)\} < \exp(G)$. For $r = 1$, the sequence $U = 0$ has $k(U) = 1 = K^*(C_{p^{m_r}}) = K(C_{p^{m_r}})$. \qed

Corollary 3.13. Suppose that $G$ is not exceptional and let $U \in \mathcal{A}(G)$ with $k(U) = K(G)$. If $g \in \text{supp}(U)$ such that $\text{ord}(h) | \text{ord}(g)$ for all $h \in \text{supp}(U)$, then $\text{ord}(g) = \exp(G)$.

Proof. If $G$ is a $p$-group, then the assertion follows from Corollary 3.12. Therefore we may assume $G$ is not a $p$-group, and we also assume to the contrary that $\text{ord}(g) < \exp(G)$. Then there exists some $p \in \mathbb{P}$ such that $\alpha = \nu_p(\text{ord}(g)) < \nu_p(\exp(G))$. Thus, since $\text{ord}(h) | \text{ord}(g)$), it follows that $\nu_p(\text{ord}(h)) \leq \alpha$ for all $h \in \text{supp}(U)$. By Theorem 3.7, $(\text{supp}(U))$ is a $p$-group if and only if $G$ is a $p$-group. Therefore $(\text{supp}(U))$ is not a $p$-group, whence $\text{ord}(g)$ is not a power of $p$. Thus $\text{ord}(g) = p^0 t$ for some $t \geq 2$. We pick some $g_0 \in G$ with $\text{ord}(g_0) = p^{\alpha+1}$ and set $g' = g - (p-1)g_0$. Then $\text{ord}(g') = p^{\alpha+1}t$ and

$$U' = g_0^{-1} g' U g^{-1} \in \mathcal{A}(G)$$

with

$$k(U') - k(U) = \frac{p-1}{p^{\alpha+1}} + \frac{1}{tp^{\alpha+1}} - \frac{1}{tp^\alpha} = \frac{(t-1)(p-1)}{tp^{\alpha+1}} > 0,$$

contradicting $k(U) = K(G)$. \qed

Theorem 3.14. Let $G = G_1 \oplus \ldots \oplus G_s$, where $s \geq 2$ and $G_1, \ldots, G_s$ are the non-trivial primary components of $G$. For $V \in \mathcal{A}(G)$, we set $\theta(V) = |\{g \in \text{supp}(V) \mid \text{ord}(g) \text{ is not a prime power}\}|$. Then the following statements are equivalent:

(a) $K(G) = K^*(G)$.

(b) For every $V \in \mathcal{A}(G)$ with $\theta(V) > 1$, there exists some $U \in \mathcal{A}(G)$ with $k(V) \leq k(U)$ and $\theta(U) < \theta(V)$.

(c) There exists some $U \in \mathcal{A}(G)$ with $k(U) = K(G)$ such that

$$U = g \prod_{i=1}^s U_i, \quad \text{where} \quad U_i \in \mathcal{F}(G_i) \quad \text{for all} \quad i \in [1, s].$$

Moreover, if $U$ has the above form, then $\text{ord}(g) = \exp(G)$ and $k(U_i) = K^*(G_i)$ for all $i \in [1, s]$.

Proof. First we show (a) implies (b). Let $(e_1, \ldots, e_s)$ be a basis of $G$ with $\text{ord}(e_i) = q_i$ a prime power for every $i \in [1, s]$. We set $g = e_1 + \ldots + e_s$. Then $\text{ord}(g) = \exp(G)$ and

$$U = g \prod_{i=1}^s e_i^{q_i-1} \in \mathcal{A}(G)$$

satisfies $k(U) = K^*(G) = K(G)$ and $\theta(U) = 1$ (since $s \geq 2$).
Next we show (b) implies (c). Condition (b) implies (since \( \theta(U) = 0 \) is impossible for \( U \in \mathcal{A}(G) \) with \( k(U) = K(G) \), in view of Theorem 3.7 and \( s \geq 2 \)) that

\[
K(G) = \max \{ k(U) \mid U \in \mathcal{A}(G) \text{ with } \theta(U) = 1 \} .
\]

Clearly, if \( U \in \mathcal{A}(G) \) with \( \theta(U) = 1 \), then \( U \) has the form given in (c).

Finally, we show (c) implies both (a) and the moreover statement that follows (c). Let \( \exp(G) = n \) and let \( p_1, \ldots, p_s \) the distinct primes which divide \( n \). For every \( i \in [1, s] \), we set

\[
\alpha_i = \max \{ v_{p_i}(\text{ord}(h)) \mid h \in \text{supp}(U_i) \} \quad \text{and} \quad \text{ord}(\sigma(U_i)) = p_i^{\beta_i} .
\]

Note that \( \beta_i \leq \alpha_i \) for every \( i \in [1, s] \). We continue with the following assertion.

**A6.** For every \( i \in [1, s] \), we have \( \beta_i = \alpha_i \).

**Proof of A6.** Let \( i \in [1, s] \). We set \( a = g + \sigma(U_i) \) with \( a \in G \), and let \( \text{ord}(a) = t \). Let \( h \in \text{supp}(U_i) \) with \( \text{ord}(h) = p_i^{\alpha_i} \) and let \( g' = a + h \). Then we have \( p_i \nmid t \) and \( t \geq 2 \) (because \( s \geq 2 \) and \( \sigma(U) = 0 \); else \( gU_i \) is a proper zero-sum subsequence, contradicting \( U \in \mathcal{A}(G) \)), \( \text{ord}(g) = tp_i^{\beta_i} \) and \( \text{ord}(g') = tp_i^{\alpha_i} \). Thus we obtain

\[
U' = g'(-\sigma(U_i))U(gh)^{-1} \in \mathcal{A}(G)
\]

and

\[
K(G) \geq k(U') = k(U) - \frac{1}{tp_i^{\beta_i}} - \frac{1}{p_i^{\alpha_i}} + \frac{1}{tp_i^{\beta_i} + 1} + \frac{1}{p_i^{\alpha_i}} = K(G) + \frac{(p_i^{\alpha_i - \beta_i} - 1)(t - 1)}{tp_i^{\alpha_i}} .
\]

This implies that \( \alpha_1 = \beta_1 \). \( \square \)

Since \( g = -(\sigma(U_1) + \ldots + \sigma(U_s)) \), A6 implies that \( \text{ord}(h) \mid \text{ord}(g) \) for all \( h \in \text{supp}(U_i) \) and \( i \in [1, s] \). Thus \( \text{ord}(g) = n \) by Corollary 3.13. Using the fact that \( k(G_i) = k^*(G_i) \) for all \( i \in [1, s] \), we obtain that

\[
K^*(G) = \frac{1}{n} + k^*(G) = \frac{1}{n} + \sum_{i=1}^{s} k^*(G_i) = \frac{1}{n} + \sum_{i=1}^{s} k(G_i)
\]

\[
\leq K(G) = k(U) = \frac{1}{n} + \sum_{i=1}^{s} k(U_i) \leq \frac{1}{n} + \sum_{i=1}^{s} k(G_i) = \frac{1}{n} + \sum_{i=1}^{s} k^*(G_i)
\]

\[
= \frac{1}{n} + k^*(G) = K^*(G) .
\]

Now all assertions follow. \( \square \)

**References**


[15]  Representation of finite abelian group elements by subsequence sums, manuscript.