

INVERSE ADDITIVE PROBLEMS FOR MINKOWSKI SUMSETS II

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ABSTRACT. The Brunn-Minkowski Theorem asserts that $\mu_d(A + B)^{1/d} \geq \mu_d(A)^{1/d} + \mu_d(B)^{1/d}$ for convex bodies $A, B \subseteq \mathbb{R}^d$, where μ_d denotes the d -dimensional Lebesgue measure. It is well-known that equality holds if and only if A and B are homothetic, but few characterizations of equality in other related bounds are known. Let H be a hyperplane. Bonnesen later strengthened this bound by showing

$$\mu_d(A + B) \geq \left(M^{1/(d-1)} + N^{1/(d-1)} \right)^{d-1} \left(\frac{\mu_d(A)}{M} + \frac{\mu_d(B)}{N} \right),$$

where $M = \sup\{\mu_{d-1}(\mathbf{x} + H \cap A) \mid \mathbf{x} \in \mathbb{R}^d\}$ and $N = \sup\{\mu_{d-1}(\mathbf{y} + H \cap B) \mid \mathbf{y} \in \mathbb{R}^d\}$. Standard compression arguments show that the above bound also holds when $M = \mu_{d-1}(\pi(A))$ and $N = \mu_{d-1}(\pi(B))$, where π denotes a projection of \mathbb{R}^d onto H , which gives an alternative generalization of the Brunn-Minkowski bound. In this paper, we characterize the cases of equality in this later bound, showing that equality holds if and only if A and B are obtained from a pair of homothetic convex bodies by ‘stretching’ along the direction of the projection, which is made formal in the paper. When $d = 2$, we characterize the case of equality in the former bound as well.

1. INTRODUCTION

Let \mathbb{R}^d denote the d -dimensional euclidian space equipped with the usual Lebesgue measure. Let $A, B \subseteq \mathbb{R}^d$ be *convex bodies*, meaning that A and B are compact, convex subsets with nonempty interior. Their Minkowski sum, or sumset, is

$$A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}.$$

Whenever the dimension of the convex body A is clear, we will use $|A|$ to denote its corresponding non-zero Lebesgue measure. For $\lambda \in \mathbb{R}$, let $\lambda A = \{\lambda \mathbf{a} \mid \mathbf{a} \in A\}$ denote the dilation of A by λ . The classical Brunn-Minkowski Theorem gives a lower bound for $|A + B|$ in terms of $|A|$ and $|B|$, and there are many far reaching generalizations and applications; see [6] for a fairly comprehensive survey. Equality is known to hold if and only if A and B are *homothetic*, that is, $A = \lambda B + \mathbf{v}$ for some $\lambda > 0$ and $\mathbf{v} \in \mathbb{R}^d$ [9, 6].

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Theorem A (Brunn-Minkowski Theorem). *If $A, B \subseteq \mathbb{R}^d$ are convex bodies, then*

$$(1) \quad |A + B| \geq \left(|A|^{1/d} + |B|^{1/d} \right)^d.$$

For $M, N > 0$, it can be shown (as remarked in [4, 6]) that

$$(2) \quad \left(M^{1/(d-1)} + N^{1/(d-1)} \right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N} \right) \geq \left(|A|^{1/d} + |B|^{1/d} \right)^d,$$

with equality only when

$$M|B|^{\frac{d-1}{d}} = N|A|^{\frac{d-1}{d}}.$$

Consequently, the following result given by Bonnesen in 1929 (see e.g. [1, 2, 4, 6]) improves the Brunn-Minkowski Inequality. Note, since A and B are compact with nonempty interiors, that the values M and N in Theorem B are nonzero and actually attained for some $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}^d$. For $d = 1$, the coefficients of $|A|$ and $|B|$ in Bonnesen's Bound are to be interpreted as their natural limiting values, i.e., $|A + B| \geq |A| + |B|$.

Theorem B (Bonnesen's Bound I). *If $A, B \subseteq \mathbb{R}^d$ are convex bodies and $H \subseteq \mathbb{R}^d$ is a $(d - 1)$ -dimensional subspace, then*

$$(3) \quad |A + B| \geq \left(M^{1/(d-1)} + N^{1/(d-1)} \right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N} \right),$$

where $M = \sup\{|(\mathbf{x} + H) \cap A| \mid \mathbf{x} \in \mathbb{R}^d\}$ and $N = \sup\{|(\mathbf{y} + H) \cap B| \mid \mathbf{y} \in \mathbb{R}^d\}$.

By standard symmetrization or compression arguments (see e.g. [10, 8] or the proof of Lemma 2.1), Theorem B implies the following alternative generalization of the Brunn-Minkowski Theorem.

Theorem C (Bonnesen's Bound II). *If $A, B \subseteq \mathbb{R}^d$ are convex bodies and $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear transformation with $\dim(\ker \pi) = 1$, then*

$$(4) \quad |A + B| \geq \left(M^{1/(d-1)} + N^{1/(d-1)} \right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N} \right),$$

where $M = |\pi(A)|$ and $N = |\pi(B)|$.

In fact, Theorems A, B and C remain true for any subsets $A, B \subseteq \mathbb{R}^d$ such that all involved quantities are measurable (see [9]). However, the general measurable case is rather painful from a technical point of view, and it is a rare textbook that is willing to reproduce the full proof of the case of inequality in Theorem A for measurable subsets. To avoid similar issues and present our ideas with greater clarity, we have focused here only on the case of convex bodies. The formulation given in Theorem C actually arises naturally when attempting to give a discrete version of the Brunn-Minkowski Theorem valid in \mathbb{Z}^d ; see [8, 5], or [7] for a discrete version of a somewhat different form.

We will use the following notation throughout the paper. Let $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a linear transformation with $\dim(\ker \pi) = 1$. Then $\pi(\mathbb{R}^d) = K$ for some $(d-1)$ -dimensional subspace K . Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{d-1} \in \mathbb{R}^d$ be an orthonormal basis for \mathbb{R}^d such that $\mathbf{e}_1, \dots, \mathbf{e}_{d-1}$ span K . Since $\dim(\ker \pi) = 1$, we have $\ker \pi = \mathbb{R}\mathbf{u}$ for any nonzero $\mathbf{u} \in \ker \pi$. Choose $\mathbf{u} \in \ker \pi$ such that the elements $\mathbf{u}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}$ form a basis for \mathbb{R}^d with the linear isomorphism $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by $\varphi(\mathbf{e}_i) = \mathbf{e}_i$ for $i \geq 1$ and $\varphi(\mathbf{e}_0) = \mathbf{u}$ being volume preserving.

Then an element $\mathbf{x} = x_0\mathbf{u} + x_1\mathbf{e}_1 + \dots + x_{d-1}\mathbf{e}_{d-1} \in \mathbb{R}^d$ may be written as $\mathbf{x} = (x_0, x_1, \dots, x_{d-1})$ and a convex body $A \subseteq \mathbb{R}^d$ can be described as

$$(5) \quad A = \{(y, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} \mid \mathbf{x} \in \pi(A), u_A(\mathbf{x}) \leq y \leq v_A(\mathbf{x})\}$$

with $u_A : \pi(A) \subseteq \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ a convex function and $v_A : \pi(A) \subseteq \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ a concave function. We say that A' is a *stretching* of A (with respect to π) of amount $h \geq 0$ if

$$A' = \{(y, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} \mid \mathbf{x} \in \pi(A), u_A(\mathbf{x}) \leq y \leq v_A(\mathbf{x}) + h\}.$$

When $\mathbf{u} = \mathbf{e}_0$, which we will be able to assume as a normalization condition as explained at the beginning of Section 2, we speak of a *vertical stretching*.

The goal of this paper is to characterize the pairs A and B for which equality holds in Theorem C.

Theorem 1.1. *Let $A, B \subseteq \mathbb{R}^d$ be convex bodies and let $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a linear transformation with $\dim(\ker \pi) = 1$. Then*

$$(6) \quad |A + B| = \left(M^{1/(d-1)} + N^{1/(d-1)}\right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N}\right),$$

where $M = |\pi(A)|$ and $N = |\pi(B)|$, if and only if there are homothetic convex bodies $A', B' \subseteq \mathbb{R}^d$ such that A is a stretching of A' and B is a stretching of B' , both with respect to π .

When $d = 2$, we also give a simple argument to derive the characterization of equality in Theorem B from the characterization of equality in Theorem C.

Theorem 1.2. *Let $H \subseteq \mathbb{R}^2$ be a one dimensional subspace and let $A, B \subseteq \mathbb{R}^2$ be convex bodies translated so that*

$$M := |H \cap A| = \sup\{|(\mathbf{x} + H) \cap A| \mid \mathbf{x} \in \mathbb{R}^2\} \quad \text{and}$$

$$N := |H \cap B| = \sup\{|(\mathbf{x} + H) \cap B| \mid \mathbf{x} \in \mathbb{R}^2\}.$$

Then

$$(7) \quad |A + B| = (M + N) \left(\frac{|A|}{M} + \frac{|B|}{N}\right)$$

if and only if there exists a linear transformation $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and homothetic convex bodies $A', B' \subseteq \mathbb{R}^2$ such that $\pi(\mathbb{R}^2) = H$,

$$\pi(A) = \pi(A') = H \cap A = H \cap A' \quad \text{and} \quad \pi(B) = \pi(B') = H \cap B = H \cap B'$$

with A a stretching of A' and B a stretching of B' , both with respect to π .

2. EQUALITY IN THE PROJECTION BONNESEN BOUND

The goal of this section is to prove Theorem 1.1. The case $d = 1$ is trivial, so we henceforth assume $d \geq 2$. We use the notation introduced before Theorem 1.1. Then, letting $\pi' : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denote the projection given by $\pi'(y_0 \mathbf{e}_0 + y_1 \mathbf{e}_1 + \dots + y_{d-1} \mathbf{e}_{d-1}) = y_1 \mathbf{e}_1 + \dots + y_{d-1} \mathbf{e}_{d-1}$, we have $|\pi(A)| = |\pi'(\varphi^{-1}(A))|$ and $|\pi(B)| = |\pi'(\varphi^{-1}(B))|$. Since φ is volume preserving, and hence φ^{-1} as well, we see that it suffices to prove the theorem when $\mathbf{u} = \mathbf{e}_0$, as we can then apply this case of the theorem to $\varphi^{-1}(A) + \varphi^{-1}(B)$, derive the structure of $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$, and then find the structure of A and B by applying the linear isomorphism φ . Thus we assume $\mathbf{u} = \mathbf{e}_0$ throughout this section. In particular, $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes the projection given by

$$\pi(x_0, x_1, \dots, x_{d-1}) = (0, x_1, \dots, x_{d-1}).$$

The proof requires a solid grasp of the fundamental metric properties and differential calculus of convex functions; see, e.g., [10, 12, 3]. We summarize the needed points below for the convenience of the reader.

2.1. Convex Calculus Basics. If $S \subseteq \mathbb{R}^{d-1}$ is a convex set and $f : S \rightarrow \mathbb{R}_{\leq 0}$, then we let

$$\text{epi}^* f = \{(y, \mathbf{x}) \mid y \in \mathbb{R}, \mathbf{x} \in S, f(\mathbf{x}) \leq y \leq 0\} \subseteq \mathbb{R}^d$$

denote the (truncated) epigraph of f in \mathbb{R}^d . Following the standard convention in the theory of convex analysis, the above definition of epigraph is written *upside down*. This is done, in part, because under this convention, the function f is convex precisely when $\text{epi}^* f$ is a convex set.

Recall that a function $f : S \rightarrow \mathbb{R}_{\geq 0}$ is *concave* if and only if $-f$ is convex, which is equivalent to

$$-\text{epi}^*(-f) = \{(y, \mathbf{x}) \mid y \in \mathbb{R}, \mathbf{x} \in S, 0 \leq y \leq f(\mathbf{x})\} \subseteq \mathbb{R}^d$$

being convex. For $\mathbf{z} \in \mathbb{R}^{d-1}$, we let

$$f'(\mathbf{x}; \mathbf{z}) := \lim_{\substack{\lambda \rightarrow 0 \\ \lambda > 0}} \frac{f(\mathbf{x} + \lambda \mathbf{z}) - f(\mathbf{x})}{\lambda}$$

denote the *onesided directional derivative* of f at \mathbf{x} with respect to the direction \mathbf{z} , and then

$$-f'(\mathbf{x}; -\mathbf{z}) = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda < 0}} \frac{f(\mathbf{x} + \lambda \mathbf{z}) - f(\mathbf{x})}{\lambda}.$$

When $d = 2$, there are only two directions, and $f_+(x) := f'(x; 1)$ is called the *right derivative* and $f_-(x) := -f'(x; -1)$ the *left derivative*. It is a basic property of convex functions that

$$\frac{f(\mathbf{x} + \lambda) - f(\mathbf{x})}{\lambda},$$

for $\lambda > 0$, is a non-decreasing function of λ (and thus a non-increasing function of $\lambda > 0$ for concave functions f), so that $f'(\mathbf{x}; \mathbf{z})$ always exists (apart from points on the boundary of S where $f(\mathbf{x} + \lambda\mathbf{z})$ is undefined for all $\lambda > 0$). Moreover, $-f'(\mathbf{x}; -\mathbf{z}) \leq f'(\mathbf{x}; \mathbf{z})$ with equality occurring precisely when f is differentiable at \mathbf{x} in the direction \mathbf{z} , in which case the usual derivative is equal to $-f'(\mathbf{x}; -\mathbf{z}) = f'(\mathbf{x}; \mathbf{z})$.

At a differentiable point $\mathbf{x} \in \text{int } S \subseteq \mathbb{R}^{d-1}$, where $\text{int } S$ denotes the interior of S , there is a unique tangent hyperplane passing through $(f(\mathbf{x}), \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$, which gives rise to the usual gradient $\nabla f(\mathbf{x}) \in \mathbb{R}^{d-1}$, whose i -th coordinate is the usual derivative $f'(\mathbf{x}; \mathbf{e}_i)$. When f is not differentiable at \mathbf{x} , there is not a unique tangent hyperplane passing through $(f(\mathbf{x}), \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Instead, there are several supporting hyperplanes passing through $(f(\mathbf{x}), \mathbf{x})$, each one giving rise to a different subgradient at \mathbf{x} . We let $\partial f(\mathbf{x})$ be the subdifferential of f at \mathbf{x} , which is the set of all subgradients $\mathbf{x}^* \in \mathbb{R}^{d-1}$, formally, all $\mathbf{x}^* \in \mathbb{R}^{d-1}$ such that the graph of the affine function $h(\mathbf{z}) = f(\mathbf{x}) + \langle \mathbf{x}^*, \mathbf{z} - \mathbf{x} \rangle$ is a non-vertical supporting hyperplane to $\text{epi } f$ at $(f(\mathbf{x}), \mathbf{x})$, which can be alternatively phrased as all $\mathbf{x}^* \in \mathbb{R}^{d-1}$ such that

$$f(\mathbf{z}) \geq f(\mathbf{x}) + \langle \mathbf{x}^*, \mathbf{z} - \mathbf{x} \rangle \quad \text{for all } \mathbf{z} \in \mathbb{R}^{d-1}.$$

When $d = 2$, this is simply the set $\partial f(x) = [f_-(x), f_+(x)]$ consisting of all possible slopes of a tangent line passing through $(f(x), x)$. For instance, if $f(x) = |x| - C$, then $\partial f(0) = [-1, 1]$, $\partial f(x) = \{-1\}$ for $x < 0$, and $\partial f(x) = \{1\}$ for $x > 0$.

When f is convex, it is differentiable a.e. with f' continuous on the subset of points where it is defined. In fact, f is Lipschitz continuous in each variable, and thus absolutely continuous, so that the Fundamental Theorem of Calculus holds. In particular, if all partial derivatives are zero a.e., then f must be a constant function. The subdifferential is continuous in the sense that, given any point \mathbf{x} in the interior of the domain of f and any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(8) \quad \partial f(\mathbf{z}) \subseteq \partial f(\mathbf{x}) + \mathcal{B}_\epsilon \quad \text{for all } \mathbf{z} \in \mathbf{x} + \mathcal{B}_\delta,$$

where \mathcal{B}_ρ denotes an open ball of radius ρ (see e.g. [10, Corollary 24.5.1].) With regards to minimizing a convex function, we have the rather striking property that a point \mathbf{x} is a global minimum for a convex function f if and only if \mathbf{x} is a local minimum, which occurs precisely when $\mathbf{0} \in \partial f(\mathbf{x})$ (see e.g. [10, Section 27].)

For a subset $A \subseteq \mathbb{R}^d$ and $\lambda \geq 0$, we let $(A)_\lambda = \bigcup_{\mathbf{x} \in A} (\mathbf{x} + \mathcal{B}_\lambda)$ denote the neighborhood of A consisting of all points strictly within distance λ from a point of A . Then the *Hausdorff*

distance between two sets $A, B \subseteq \mathbb{R}^d$ is defined as

$$d_H(A, B) := \inf\{\lambda \geq 0 \mid A \subseteq (B)_\lambda \text{ and } B \subseteq (A)_\lambda\}.$$

When restricted to closed subsets of \mathbb{R}^d , $d_H(\cdot, \cdot)$ becomes a metric; in particular, $d_H(A, B) = 0$, for closed subsets $A, B \subseteq \mathbb{R}^d$, if and only if $A = B$. Blaschke's Theorem (see e.g. [3, 12]) asserts that the Hausdorff metric space is compact when restricted to convex bodies all contained within some fixed closed ball in \mathbb{R}^d . In particular, if $A_1 \subseteq A_2 \subseteq \dots$ is an increasing sequence of convex bodies all contained within some fixed closed ball in \mathbb{R}^d , then $A_i \rightarrow A$, where A is the closure of $\bigcup_{i \geq 1} A_i$ and the limit is with respect to the Hausdorff metric. Additionally, the limit of convex bodies is again convex.

2.2. A Sequence of Lemmas. Our strategy is to first prove Theorem 1.1 when A and B are the epigraphs of respective concave functions $f : S \subseteq \mathbb{R}^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ and $g : T \subseteq \mathbb{R}^{d-1} \rightarrow \mathbb{R}_{\geq 0}$, and then extend to the more general case. To do this, we break the majority of the proof into a series of lemmas. Our first lemma below allows us to restrict to the case when the domains S and T are homothetic. During the course of the proof, an outline of the proof of Theorem C is recreated.

Lemma 2.1. *Let $A, B \subseteq \mathbb{R}^d$ be convex bodies. If*

$$(9) \quad |A + B| = \left(M^{1/(d-1)} + N^{1/(d-1)} \right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N} \right),$$

where $M = |\pi(A)|$ and $N = |\pi(B)|$, then $\pi(A)$ and $\pi(B)$ are homothetic.

Proof. This is a simple consequence of compression techniques and the proof of Bonnesen's Theorem as given in [4]. We outline the details here. Recalling that we have assumed $\mathbf{u} = \mathbf{e}_0$ and writing a convex body A using the notation of (5), we define

$$\mathbf{C}(A) := \{(y, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} \mid \mathbf{x} \in \pi(A), 0 \leq y \leq v_A(\mathbf{x}) - u_A(\mathbf{x})\}.$$

It is easily derived (see also [8]) that

$$\begin{aligned} |\mathbf{C}(A)| &= |A| \quad \text{and} \quad |\mathbf{C}(B)| = |B|, \\ M &:= |\pi(A)| = |\pi(\mathbf{C}(A))| = \sup\{|\mathbf{x} + H \cap \mathbf{C}(A)| \mid \mathbf{x} \in \mathbb{R}^d\}, \\ N &:= |\pi(B)| = |\pi(\mathbf{C}(B))| = \sup\{|\mathbf{x} + H \cap \mathbf{C}(B)| \mid \mathbf{x} \in \mathbb{R}^d\}, \\ |A + B| &\geq |\mathbf{C}(A) + \mathbf{C}(B)|, \end{aligned}$$

where $H = \mathbf{e}_0^\perp$ is the orthogonal space to \mathbf{e}_0 , which is spanned by $\mathbf{e}_1, \dots, \mathbf{e}_{d-1}$. For $z \in \mathbb{R}$, let $A(z) = \mathbf{C}(A) \cap (z\mathbf{e}_0 + H)$ and $B(z) = \mathbf{C}(B) \cap (z\mathbf{e}_0 + H)$. Then

$$\begin{aligned} |\mathbf{C}(A) + \mathbf{C}(B)| &\geq \int_{-\infty}^{+\infty} \sup_{x+y=z} \{|A(x) + B(y)|\} dz \\ (10) \qquad \qquad &\geq \int_{-\infty}^{+\infty} \sup_{x+y=z} \left\{ \left(|A(x)|^{1/(d-1)} + |B(y)|^{1/(d-1)} \right)^{d-1} \right\} dz \end{aligned}$$

$$(11) \qquad \qquad \geq \left(M^{1/(d-1)} + N^{1/(d-1)} \right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N} \right),$$

where (10) follows by the Brunn-Minkowski Theorem applied to each $A(x) + B(y)$, and (11) follows by [4, Theorem 2.1] (as in the proof of Bonnesen's Bound given in [4]). Consequently, in view of (9), we see that equality must hold in (10). The remainder of the proof now follows easily from the following two basic claims concerning convex bodies.

Claim 1. If $X, Y \subseteq \mathbb{R}^{d-1}$ are convex bodies that are not homothetic, then there exists $\delta > 0$ such that no two convex bodies $C, D \subseteq \mathbb{R}^{d-1}$ with $\mathbf{d}_H(X, C) < \delta$ and $\mathbf{d}_H(Y, D) < \delta$ are homothetic.

Proof. If the claim is false, then there exist two sequences of convex bodies $\{C_i\}_{i \geq 1}$ and $\{D_i\}_{i \geq 1}$ such that $C_i \rightarrow X$, $D_i \rightarrow Y$ and, for each $i \geq 1$, C_i and D_i are homothetic, so that $D_i = \alpha_i C_i + \mathbf{x}_i$ for some $\alpha_i > 0$ and $\mathbf{x}_i \in \mathbb{R}^{d-1}$. Since each of the sequences $\{C_i\}_{i \geq 1}$ and $\{\alpha_i C_i + \mathbf{x}_i\}_{i \geq 1}$ converges to a convex body, it is easily verified that $\alpha_i \rightarrow \alpha$ and $\mathbf{x}_i \rightarrow \mathbf{x}$ for some $\alpha > 0$ and $\mathbf{x} \in \mathbb{R}^{d-1}$. Hence $Y = \alpha X + \mathbf{x}$, contrary to the hypothesis. \square

Claim 2. For any $\delta > 0$, there exists an $\epsilon > 0$ such that $\mathbf{d}_H(A(0), A(x)) < \delta$ and $\mathbf{d}_H(B(0), B(y)) < \delta$ for all $x, y \in [0, \epsilon)$.

Proof. If the claim fails for (say) A , then we can find a sequence $x_1 > x_2 > \dots$, where $x_i \in \mathbb{R}_{>0}$, such that $x_i \rightarrow 0$ and $\mathbf{d}_H(A(0), A(x_i)) \geq \delta > 0$ for all i . Since $x_1 > x_2 > \dots$, it follows from the definition of $A(x)$ that $A(x_1) \subseteq A(x_2) \subseteq \dots$. Thus $A(x_i) \rightarrow A'$, where A' is the closure of $\bigcup_{i \geq 1} A(x_i)$. Since $x_i \rightarrow 0$ with $x_i > 0$, it follows that $\bigcup_{i \geq 1} A(x_i)$ consist of all points $\mathbf{x} \in \mathbb{R}^{d-1}$ such that $(y, \mathbf{x}) \in \mathbf{C}(A)$ for some $y > 0$. Consequently, as $\mathbf{C}(A) = -\text{epi}^*(u_A - v_A)$ is a convex body (both $-v_A$ and u_A are convex functions), so that $u_A - v_A \leq 0$ cannot be the constant zero function, it follows by a simple argument that $\text{int}(A(0)) \subseteq \bigcup_{i \geq 1} A(x_i)$, whence $A' = A(0)$. But since $A(x_i) \rightarrow A' = A(0)$, it now follows that $\mathbf{d}_H(A(x_i), A(0)) \rightarrow 0$, contradicting that $\mathbf{d}_H(A(0), A(x_i)) \geq \delta > 0$ for all i . This completes the claim. \square

We now complete the proof the Lemma. If by contradiction $A(0) = \pi(A)$ and $B(0) = \pi(B)$ are not homothetic, then, by Claims 1 and 2 (take $X = A(0)$ and $Y = B(0)$ in Claim

1 to find the δ to be used for Claim 2), there is some $\epsilon > 0$ such that $A(x)$ and $B(y)$ are not homothetic for all $x, y \in [0, \epsilon)$. As a result, the application of the Brunn-Minkowski Theorem to (10) yielded a strict inequality for all for $z \in [0, \epsilon)$, whence equality in (10) is impossible, contrary to our assumption. \square

The following lemma shows that vertical stretching preserves equality (6) provided $\pi(A)$ and $\pi(B)$ are homothetic, which we will be able to assume using Lemma 2.1. Not only does this show that the sets described by Theorem 1.1 satisfy the equality (6), but it will also play an important role in the other direction of the proof of Theorem 1.1, allowing us to consider convex bodies sufficiently stretched and thereby resolve a delicate technical difficulty with ease.

Lemma 2.2. *Let $A, B, A', B' \subseteq \mathbb{R}^d$ be convex bodies and suppose that A and B are vertical stretchings of A' and B' , respectively. Then*

$$\begin{aligned} |A + B| - \left(M^{1/(d-1)} + N^{1/(d-1)}\right)^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N}\right) &\geq \\ |A' + B'| - \left(M^{1/(d-1)} + N^{1/(d-1)}\right)^{d-1} \left(\frac{|A'|}{M} + \frac{|B'|}{N}\right), & \end{aligned}$$

where $M = |\pi(A)| = |\pi(A')|$ and $N = |\pi(B)| = |\pi(B')|$, with equality if and only if $\pi(A) = \pi(A')$ and $\pi(B) = \pi(B')$ are homothetic.

Proof. Suppose that A is a stretching of A' of amount α and B is a stretching of B' of amount β , where $\alpha \geq 0$ and $\beta \geq 0$. Then

$$(12) \quad |A| = |A'| + |\pi(A')|\alpha = |A'| + M\alpha \quad \text{and}$$

$$(13) \quad |B| = |B'| + |\pi(B')|\beta = |B'| + N\beta.$$

For $\mathbf{z} \in \pi(A + B)$, observe that

$$(14) \quad \pi^{-1}(\mathbf{z}) \cap (A + B) = \bigcup_{\mathbf{x}+\mathbf{y}=\mathbf{z}} ((\pi^{-1}(\mathbf{x}) \cap A) + (\pi^{-1}(\mathbf{y}) \cap B)).$$

Both $\pi^{-1}(\mathbf{x}) \cap A$ and $\pi^{-1}(\mathbf{y}) \cap B$ are vertical line segments (as A and B are convex). Moreover, since $A + B$ is also convex, their union in (14) must again be a vertical line segment. The vertical line segment $\pi^{-1}(\mathbf{x}) \cap A$ is obtained by extending the line segment $\pi^{-1}(\mathbf{x}) \cap A'$ by an additional length of α appended onto the top of the segment $\pi^{-1}(\mathbf{x}) \cap A'$; the line segment $\pi^{-1}(\mathbf{x}) \cap B$ is likewise obtained from $\pi^{-1}(\mathbf{x}) \cap B'$ appending on an additional length of β to the top of $\pi^{-1}(\mathbf{x}) \cap B'$. Thus, since the union in (14) is a single vertical line segment, it follows that

$$\left| \bigcup_{\mathbf{x}+\mathbf{y}=\mathbf{z}} ((\pi^{-1}(\mathbf{x}) \cap A) + (\pi^{-1}(\mathbf{y}) \cap B)) \right| = \left| \bigcup_{\mathbf{x}+\mathbf{y}=\mathbf{z}} ((\pi^{-1}(\mathbf{x}) \cap A') + (\pi^{-1}(\mathbf{y}) \cap B')) \right| + (\alpha + \beta)$$

for each $\mathbf{z} \in \pi(A + B) = \pi(A' + B') = \pi(A') + \pi(B')$. Consequently,

$$(15) \quad \begin{aligned} |A + B| &= |A' + B'| + |\pi(A') + \pi(B')|(\alpha + \beta) \\ &\geq |A' + B'| + (M^{1/(d-1)} + N^{1/(d-1)})^{d-1}(\alpha + \beta), \end{aligned}$$

where (15) is obtained by applying the Brunn-Minkowski Theorem, with equality if and only if $\pi(A) = \pi(A')$ and $\pi(B) = \pi(B')$ are homothetic.

From (15), we conclude that

$$(16) \quad \begin{aligned} |A + B| - |A' + B'| &= |\pi(A') + \pi(B')|(\alpha + \beta) \\ &\geq \left(M^{1/(d-1)} + N^{1/(d-1)}\right)^{d-1} (\alpha + \beta), \end{aligned}$$

with equality if and only if $\pi(A) = \pi(A')$ and $\pi(B) = \pi(B')$ are homothetic. Also, (12) and (13) yield

$$(17) \quad \begin{aligned} &\left(M^{1/(d-1)} + N^{1/(d-1)}\right)^{d-1} \left(\left(\frac{|A|}{M} + \frac{|B|}{N}\right) - \left(\frac{|A'|}{M} + \frac{|B'|}{N}\right) \right) = \\ &\left(M^{1/(d-1)} + N^{1/(d-1)}\right)^{d-1} (\alpha + \beta). \end{aligned}$$

Comparing (16) and (17) completes the lemma. \square

Lemma 2.3 provides the base case for the inductive proof of Lemma 2.4, which will be our main argument, combined with standard approximation arguments, for characterizing the case of equality in Bonnesen's Bound for epigraphs.

Lemma 2.3. *Let $m, n > 0$ and let $f : [0, m] \rightarrow \mathbb{R}_{\geq 0}$ and $g : [0, n] \rightarrow \mathbb{R}_{\geq 0}$ be concave functions. Let $A, B \subseteq \mathbb{R}^2$ be defined as $A = -\text{epi}^*(-f)$ and $B = -\text{epi}^*(-g)$.*

(a) *Then*

$$(18) \quad |A + B| \geq (m + n) \left(\frac{|A|}{m} + \frac{|B|}{n} \right) + \Delta,$$

where

$$\Delta = \left(nf(m) - \frac{n}{m} \int_0^m f(x) dx \right) + \left(mg(0) - \frac{m}{n} \int_0^n g(x) dx \right).$$

(b) *In particular, if $f'_+(x) \geq g'_+(y) + \epsilon$ for all $x \in [0, m]$ and $y \in [0, n]$, where $\epsilon \geq 0$, then*

$$|A + B| \geq (m + n) \left(\frac{|A|}{m} + \frac{|B|}{n} \right) + \frac{mn}{2} \epsilon.$$

Proof. We first observe that

$$\begin{aligned} A + ([0, g(0)] \times \{0\}) &\subseteq (A + B) \cap (\mathbb{R} \times [0, m]), \\ ([0, f(m)] \times \{m\}) + B &\subseteq (A + B) \cap (\mathbb{R} \times [m, m + n]). \end{aligned}$$

Therefore,

$$\begin{aligned} |A + B| &\geq |A + ([0, g(0)] \times \{0\})| + |([0, f(m)] \times \{m\}) + B| \\ &= (|A| + mg(0)) + (|B| + nf(m)), \end{aligned}$$

and we complete the proof of part (a) as follows:

$$\begin{aligned} |A + B| - (m + n)\left(\frac{|A|}{m} + \frac{|B|}{n}\right) &\geq |A| + mg(0) + |B| + nf(m) - (m + n)\left(\frac{|A|}{m} + \frac{|B|}{n}\right) \\ &= mg(0) + nf(m) - \frac{n}{m}|A| - \frac{m}{n}|B| \\ &= n\left(f(m) - \frac{|A|}{m}\right) + m\left(g(0) - \frac{|B|}{n}\right) = \Delta. \end{aligned}$$

It remains to prove part (b). Thus suppose $f'_+(x) \geq g'_+(y) + \epsilon$ for all $x \in [0, m]$ and $y \in [0, n]$, where $\epsilon \geq 0$. The product of absolutely continuous functions defined over a closed, bounded interval is absolutely continuous on this interval. Thus, since $f : [0, m] \rightarrow \mathbb{R}_{\geq 0}$ is a concave function, and thus absolutely continuous (and hence differentiable a.e.), it follows that $xf(x) : [0, m] \rightarrow \mathbb{R}_{\geq 0}$ is also absolutely continuous (and hence differentiable a.e.). As a result, noting that $(xf(x))' = f(x) + xf'(x)$ a.e., it follows from the Fundamental Theorem of Calculus that

$$f(m) = \frac{1}{m} \int_0^m (xf(x))' dx = \frac{1}{m} \int_0^m xf'(x) dx + \frac{1}{m} \int_0^m f(x) dx.$$

Hence we may rewrite Δ as

$$\Delta = \frac{n}{m} \int_0^m yf'(y) dy - \frac{m}{n} \int_0^n g(x) dx + mg(0).$$

Applying the substitution $y \mapsto \frac{m}{n}x$ to the first integral and using the fact that $f'(x) = f'_+(x)$ a.e., we obtain

$$(19) \quad \Delta = mg(0) + \frac{m}{n} \int_0^n \left(xf'_+\left(\frac{m}{n}x\right) - g(x)\right) dx.$$

Since $f'_+(x) \geq g'_+(y) + \epsilon$ for all $x \in [0, m]$ and $y \in [0, n]$, it follows that

$$(20) \quad xf'_+\left(\frac{m}{n}x\right) \geq xg'_+(0) + x\epsilon,$$

for all $x \in [0, n]$. Since g is concave, $\frac{g(x) - g(0)}{x}$ is a non-increasing function of x , whence

$$(21) \quad g'_+(0) = \lim_{\substack{\lambda \rightarrow 0 \\ \lambda > 0}} \frac{g(\lambda) - g(0)}{\lambda} \geq \frac{g(x) - g(0)}{x}$$

for all $x \in (0, n)$. Applying the estimates (20) and (21) to (19), we obtain

$$\begin{aligned}
 \Delta &\geq mg(0) + \frac{m}{n} \int_0^n (xg'_+(0) + x\epsilon - g(x)) \, dx \\
 (22) \quad &\geq mg(0) + \frac{m}{n} \int_0^n (-g(0) + x\epsilon) \, dx = \frac{mn}{2}\epsilon,
 \end{aligned}$$

which combined with (18) implies the desired bound. \square

The proof of the following lemma essentially contains a proof of Theorem C for $d \geq 3$ using the case $d = 2$ as the base of an inductive argument. The inductive application of Theorem C is used to make a kind of $(d - 2)$ -dimensional compression possible.

Lemma 2.4. *Let $d \geq 2$, let $m, n > 0$ and let $f : [0, m]^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ and $g : [0, n]^{d-1} \rightarrow \mathbb{R}_{\geq 0}$ be concave functions. Let $A, B \subseteq \mathbb{R}^d$ be defined as $A = -\text{epi}^*(-f)$ and $B = -\text{epi}^*(-g)$. Suppose*

$$f'(\mathbf{x}; \mathbf{e}_1) \geq g'(\mathbf{y}; \mathbf{e}_1) + \epsilon \quad \text{for all } \mathbf{x} \in [0, m]^{d-1} \text{ and } \mathbf{y} \in [0, n]^{d-1},$$

where $\epsilon \geq 0$. Then

$$|A + B| \geq (m + n)^{d-1} \left(\frac{|A|}{m^{d-1}} + \frac{|B|}{n^{d-1}} \right) + \frac{mn}{2}(m + n)^{d-2}\epsilon.$$

Proof. When $d = 2$, Lemma 2.3 yields the desired bound. We assume $d \geq 3$ and proceed by induction on d . For $x \in [0, m]$ and $y \in [0, n]$, let $f_x : [0, m]^{d-2} \rightarrow \mathbb{R}_{\geq 0}$ and $g_y : [0, n]^{d-2} \rightarrow \mathbb{R}_{\geq 0}$ be defined by $f_x(x_1, \dots, x_{d-2}) = f(x_1, \dots, x_{d-2}, x)$ and $g_y(y_1, \dots, y_{d-2}) = g(y_1, \dots, y_{d-2}, y)$. Then $-\text{epi}^*(-f_x) = (\mathbb{R}^{d-1} \times \{x\}) \cap A$ is the x -section of A , and we will denote this set by $A(x) = (\mathbb{R}^{d-1} \times \{x\}) \cap A$. Likewise define $B(y) = -\text{epi}^*(-g_y) = (\mathbb{R}^{d-1} \times \{y\}) \cap B$ and, for $z \in [0, m + n]$, let $(A + B)(z) = (\mathbb{R}^{d-1} \times \{z\}) \cap (A + B)$. Then $(A + B)(z) = \bigcup_{x+y=z} (A(x) + B(y))$. Consequently,

$$(23) \quad |A + B| \geq \int_0^{m+n} \sup_{x+y=z} \{|A(x) + B(y)|\} \, dz.$$

By induction hypothesis, we know

$$|A(x) + B(y)| \geq (m + n)^{d-2} \left(\frac{|A(x)|}{m^{d-2}} + \frac{|B(y)|}{n^{d-2}} \right) + \frac{mn}{2}(m + n)^{d-3}\epsilon,$$

for all $x \in [0, m]$ and $y \in [0, n]$. Combining the above inequality with (23) gives

$$\begin{aligned}
 (24) \quad |A + B| &\geq \int_0^{m+n} \sup_{x+y=z} \left\{ \frac{(m + n)^{d-2}}{m^{d-2}} |A(x)| + \frac{(m + n)^{d-2}}{n^{d-2}} |B(y)| \right\} \, dz \\
 &\quad + \frac{mn}{2}(m + n)^{d-2}\epsilon.
 \end{aligned}$$

Let $\tilde{f} : [0, m] \rightarrow \mathbb{R}_{\geq 0}$ be the function defined by

$$\tilde{f}(x) = \frac{(m+n)^{d-2}}{m^{d-2}} |A(x)|$$

and let $\tilde{g} : [0, n] \rightarrow \mathbb{R}_{\geq 0}$ be the function defined by

$$\tilde{g}(y) = \frac{(m+n)^{d-2}}{m^{d-2}} |B(y)|.$$

Let $\tilde{A} = -\text{epi}^*(-\tilde{f})$ and $\tilde{B} = -\text{epi}^*(-\tilde{g})$. As A and B are convex bodies, the functions $|A(x)|$ and $|B(y)|$ are both integrable, and thus \tilde{f} and \tilde{g} as well. Moreover,

$$\begin{aligned} |\tilde{A} + \tilde{B}| &= \int_0^{m+n} \sup_{x+y=z} \{\tilde{f}(x) + \tilde{g}(y)\} dz \\ (25) \quad &= \int_0^{m+n} \sup_{x+y=z} \left\{ \frac{(m+n)^{d-2}}{m^{d-2}} |A(x)| + \frac{(m+n)^{d-2}}{m^{d-2}} |B(y)| \right\} dz. \end{aligned}$$

Applying Theorem C, which (as mentioned in the introduction) holds more generally for any compact subsets A and B , to $\tilde{A} + \tilde{B}$, we conclude that

$$\begin{aligned} |\tilde{A} + \tilde{B}| &\geq (m+n) \left(\frac{|\tilde{A}|}{m} + \frac{|\tilde{B}|}{n} \right) \\ &= (m+n) \left(\frac{(m+n)^{d-2}}{m^{d-1}} \int_0^m |A(x)| dx + \frac{(m+n)^{d-2}}{n^{d-1}} \int_0^n |B(y)| dy \right) \\ (26) \quad &= (m+n)^{d-1} \left(\frac{|A|}{m^{d-1}} + \frac{|B|}{n^{d-1}} \right). \end{aligned}$$

Combining (24), (25) and (26) yields the desired lower bound for $|A + B|$, completing the proof. \square

Completion of the Proof. We can now proceed with the proof of Theorem 1.1, first in the case when A and B are both epigraphs.

Lemma 2.5. *Let $d \geq 2$, let $S, T \subseteq \mathbb{R}^{d-1}$ be convex bodies, and let $f : S \rightarrow \mathbb{R}_{\geq 0}$ and $g : T \rightarrow \mathbb{R}_{\geq 0}$ be concave functions. Let $A, B \subseteq \mathbb{R}^d$ be defined as $A = -\text{epi}^*(-f)$ and $B = -\text{epi}^*(-g)$. If*

$$|A + B| = (|S|^{1/(d-1)} + |T|^{1/(d-1)})^{d-1} \left(\frac{|A|}{|S|} + \frac{|B|}{|T|} \right),$$

then S and T are homothetic and the graphs of f and g are also homothetic, i.e.,

$$f(\mathbf{x}) = \lambda g\left(\frac{1}{\lambda}(\mathbf{x} - \mathbf{x}_0)\right) + C \quad \text{for all } \mathbf{x} \in S,$$

where $C = \frac{|A|}{|S|} - \frac{\lambda|B|}{|T|}$ and $S = \lambda T + \mathbf{x}_0$ for some $\lambda > 0$ and $\mathbf{x}_0 \in \mathbb{R}^d$.

Proof. From Theorem C, we know that

$$|A + B| \geq (|S|^{1/(d-1)} + |T|^{1/(d-1)})^{d-1} \left(\frac{|A|}{|S|} + \frac{|B|}{|T|} \right).$$

We wish to characterize when equality holds. By Lemma 2.1, equality in the bound implies S and T are homothetic, say $S = \lambda T + \mathbf{x}_0$ with $\lambda > 0$ and $\mathbf{x}_0 \in \mathbb{R}^d$. Hence

$$(27) \quad |S| = \lambda^{d-1} |T|.$$

By translating appropriately, we may w.l.o.g. assume $\mathbf{x}_0 = 0$, so that $S = \lambda T$. It remains to show that the graphs of f and g are homothetic, that is, that $f(\mathbf{x}) = \lambda g(\frac{1}{\lambda}\mathbf{x}) + C$ for all $\mathbf{x} \in S$, where $C \in \mathbb{R}$ is some constant. To calculate what this constant must be, we have only to note that

$$|A| = \int_S f(\mathbf{x}) \, d\mathbf{x} = \int_S \lambda g\left(\frac{1}{\lambda}\mathbf{x}\right) + C \, d\mathbf{x} = |\lambda B| + |S|C = \lambda^d |B| + |S|C,$$

and combine this with (27), which gives $C = \frac{|A|}{|S|} - \frac{\lambda|B|}{|T|}$.

Let $\tilde{g} : S \rightarrow \mathbb{R}_{\geq 0}$ be the concave function defined by $\tilde{g}(\mathbf{x}) = \lambda g(\frac{1}{\lambda}\mathbf{x})$. Since f and g are concave functions, they are Lipschitz continuous in each variable, and thus absolutely continuous. Furthermore, $\tilde{g}'(\mathbf{x}; \mathbf{e}_j) = g'(\frac{1}{\lambda}\mathbf{x}; \mathbf{e}_j)$ for all $j \in [1, d-1]$ and all $\mathbf{x} \in S$. Consequently, if $f'(\mathbf{x}; \mathbf{e}_j) = g'(\frac{1}{\lambda}\mathbf{x}; \mathbf{e}_j) = \tilde{g}'(\mathbf{x}; \mathbf{e}_j)$ for all $j \in [1, d-1]$ and a.e. $\mathbf{x} \in \text{int}(S)$, then the Fundamental Theorem of Calculus would imply $f(\mathbf{x}) = \lambda g(\frac{1}{\lambda}\mathbf{x}) + C$ for some constant C , as desired. Therefore, if the statement of the lemma is false, then there must be some differentiable point $\mathbf{x}_0 \in S$, contained in the interior of S (as the boundary of S has measure zero), such that w.l.o.g. $f'(\mathbf{x}_0; \mathbf{e}_1) \geq g'(\frac{1}{\lambda}\mathbf{x}_0; \mathbf{e}_1) + 2\epsilon$ with $\epsilon > 0$. In view of (8), we can find a small neighborhood around $\mathbf{x}_0 \in S$ in which $f'(\mathbf{x}; \mathbf{e}_1) \geq f'(\mathbf{x}_0; \mathbf{e}_1) - \frac{\epsilon}{2}$ for \mathbf{x} in this neighborhood, as well as a small neighborhood around $\frac{1}{\lambda}\mathbf{x}_0 \in T$ in which $g'(\frac{1}{\lambda}\mathbf{x}; \mathbf{e}_1) \leq g'(\frac{1}{\lambda}\mathbf{x}_0; \mathbf{e}_1) + \frac{\epsilon}{2}$ for \mathbf{x} in this neighborhood. Restricting to smaller neighborhoods as need be, we can thus find a pair of homothetic boxes $\mathbf{x}_0 + [-\frac{1}{2}\delta, \frac{1}{2}\delta]^{d-1} \subseteq S$ and $\frac{1}{\lambda}\mathbf{x}_0 + [-\frac{1}{2\lambda}\delta, \frac{1}{2\lambda}\delta]^{d-1} \subseteq T$ such that

$$(28) \quad f'(\mathbf{x}; \mathbf{e}_1) \geq g'(\mathbf{y}; \mathbf{e}_1) + \epsilon \quad \text{for all } \mathbf{x} \in \mathbf{x}_0 + [-\frac{1}{2}\delta, \frac{1}{2}\delta]^{d-1} \quad \text{and} \quad \mathbf{y} \in \frac{1}{\lambda}\mathbf{x}_0 + [-\frac{1}{2\lambda}\delta, \frac{1}{2\lambda}\delta]^{d-1},$$

where $\epsilon > 0$.

The remainder of the argument is now similar to a standard inner/outer measure approximation to evaluate a Lebesgue integrable function; see, e.g. [11]. For $k \in \{0, 1, 2, \dots\}$, partition \mathbb{R}^{d-1} into a grid using boxes of the form $\mathbf{z} + \frac{1}{2^k}[-\frac{1}{2}\delta, \frac{1}{2}\delta]^{d-1}$ such that no two boxes share an interior point and such that $\mathbf{x}_0 + [-\frac{1}{2}\delta, \frac{1}{2}\delta]^{d-1}$ is a union of some subset of these boxes. Let \mathcal{B}_k be the collection of all these boxes wholly contained in S and, for each box $\mathbf{b} \in \mathcal{B}_k$, let $A_{\mathbf{b}} \subseteq A$ be the subset $(\mathbb{R} \times \mathbf{b}) \cap A$, which corresponds to the epigraph of f restricted to the domain $\mathbf{b} \subseteq S$. Also, let $\mathcal{B}'_k \subseteq \mathcal{B}_k$ be those boxes whose union is $\frac{1}{\lambda}\mathbf{x}_0 + [-\frac{1}{2\lambda}\delta, \frac{1}{2\lambda}\delta]^{d-1}$.

Let $\frac{1}{\lambda}\mathcal{B}_k = \{\frac{1}{\lambda}\mathfrak{b} \mid \mathfrak{b} \in \mathcal{B}_k\}$. Thus $\frac{1}{\lambda}\mathcal{B}_k$ consists of boxes of the form $\mathbf{z} + \frac{1}{2^k}[-\frac{1}{2\lambda}\delta, \frac{1}{2\lambda}\delta]^{d-1}$, wholly contained in T , such that no two boxes share an interior point and such that the union of boxes from $\frac{1}{\lambda}\mathcal{B}'_k$ is equal to $\frac{1}{\lambda}\mathbf{x}_0 + [-\frac{1}{2\lambda}\delta, \frac{1}{2\lambda}\delta]^{d-1}$. For each box $\mathfrak{b} \in \mathcal{B}_k$, let $B_{\mathfrak{b}} \subseteq B$ be the subset $(\mathbb{R} \times \frac{1}{\lambda}\mathfrak{b}) \cap B$, which corresponds to the epigraph of g restricted to the domain $\frac{1}{\lambda}\mathfrak{b} \subseteq T$. Let

$$m_k = \frac{\delta}{2^k} \quad \text{and} \quad n_k = \frac{\delta}{\lambda 2^k}$$

be, respectively, the length of each side of the boxes $\mathfrak{b} \in \mathcal{B}_k$ and the length of each side of the boxes $\frac{1}{\lambda}\mathfrak{b} \in \frac{1}{\lambda}\mathcal{B}_k$. Thus

$$|\mathfrak{b}| = m_k^{d-1} \quad \text{and} \quad |\frac{1}{\lambda}\mathfrak{b}| = n_k^{d-1} \quad \text{for } \mathfrak{b} \in \mathcal{B}_k.$$

It is now easily seen that $\bigcup_{\mathfrak{b} \in \mathcal{B}_k} (A_{\mathfrak{b}} + B_{\mathfrak{b}}) \subseteq A + B$ with the intersection of any two distinct sumsets $A_{\mathfrak{b}} + B_{\mathfrak{b}}$ being a measure zero subset; of course, we can also use the more accurate estimate

$$\left(\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} A_{\mathfrak{b}} \right) + \left(\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} B_{\mathfrak{b}} \right) \subseteq A + B$$

in place of $\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} (A_{\mathfrak{b}} + B_{\mathfrak{b}})$, and its intersection with all other $A_{\mathfrak{b}} + B_{\mathfrak{b}}$, with $\mathfrak{b} \in \mathcal{B}_k \setminus \mathcal{B}'_k$, will still be a measure zero subset. Thus

$$|A + B| \geq \sum_{\mathfrak{b} \in \mathcal{B}_k \setminus \mathcal{B}'_k} |A_{\mathfrak{b}} + B_{\mathfrak{b}}| + \left| \left(\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} A_{\mathfrak{b}} \right) + \left(\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} B_{\mathfrak{b}} \right) \right|.$$

As a result, making use of (28) and applying Lemma 2.4 to

$$\left(\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} A_{\mathfrak{b}} \right) + \left(\bigcup_{\mathfrak{b} \in \mathcal{B}'_k} B_{\mathfrak{b}} \right)$$

and then using Theorem C for all other $\mathfrak{b} \in \mathcal{B}_k \setminus \mathcal{B}'_k$, we obtain

$$\begin{aligned} |A + B| &\geq \sum_{\mathfrak{b} \in \mathcal{B}_k \setminus \mathcal{B}'_k} (m_k + n_k)^{d-1} \left(\frac{|A_{\mathfrak{b}}|}{m_k^{d-1}} + \frac{|B_{\mathfrak{b}}|}{n_k^{d-1}} \right) + \\ &(m_0 + n_0)^{d-1} \left(\frac{1}{m_0^{d-1}} \sum_{\mathfrak{b} \in \mathcal{B}'_k} |A_{\mathfrak{b}}| + \frac{1}{n_0^{d-1}} \sum_{\mathfrak{b} \in \mathcal{B}'_k} |B_{\mathfrak{b}}| \right) + \frac{(m_0 n_0)}{2} (m_0 + n_0)^{d-2} \epsilon \\ &= \left(\frac{m_k + n_k}{m_k} \right)^{d-1} \sum_{\mathfrak{b} \in \mathcal{B}_k \setminus \mathcal{B}'_k} |A_{\mathfrak{b}}| + \left(\frac{m_k + n_k}{n_k} \right)^{d-1} \sum_{\mathfrak{b} \in \mathcal{B}_k \setminus \mathcal{B}'_k} |B_{\mathfrak{b}}| + \\ &\left(\frac{m_0 + n_0}{m_0} \right)^{d-1} \sum_{\mathfrak{b} \in \mathcal{B}'_k} |A_{\mathfrak{b}}| + \left(\frac{m_0 + n_0}{n_0} \right)^{d-1} \sum_{\mathfrak{b} \in \mathcal{B}'_k} |B_{\mathfrak{b}}| + \frac{\delta^d (\lambda + 1)^{d-2}}{2\lambda^{d-1}} \epsilon. \end{aligned}$$

In view of the definition of m_k and n_k , we have $\frac{m_k+n_k}{m_k} = 1 + \frac{1}{\lambda}$ and $\frac{m_k+n_k}{n_k} = 1 + \lambda$ for all $k \in \{0, 1, 2, \dots\}$. Thus the above calculation implies

$$(29) \quad |A + B| \geq \left(1 + \frac{1}{\lambda}\right)^{d-1} \sum_{\mathfrak{b} \in \mathcal{B}_k} |A_{\mathfrak{b}}| + (1 + \lambda)^{d-1} \sum_{\mathfrak{b} \in \mathcal{B}_k} |B_{\mathfrak{b}}| + \frac{\delta^d (\lambda + 1)^{d-2}}{2\lambda^{d-1}} \epsilon.$$

As $k \rightarrow \infty$, we see that $\bigcup_{\mathfrak{b} \in \mathcal{B}_k} \mathfrak{b}$ approaches S . More specifically, since S is a convex body, the difference between $\lim_{k \rightarrow \infty} \bigcup_{\mathfrak{b} \in \mathcal{B}_k} \mathfrak{b}$ and S is a measure zero subset. Since $T = \frac{1}{\lambda} S$ is just a dilation of S , we likewise see that the difference between $\lim_{k \rightarrow \infty} \bigcup_{\mathfrak{b} \in \mathcal{B}_k} \frac{1}{\lambda} \mathfrak{b}$ and T is also a measure zero subset. Consequently, $\sum_{\mathfrak{b} \in \mathcal{B}_k} |A_{\mathfrak{b}}| \rightarrow |A|$ and $\sum_{\mathfrak{b} \in \mathcal{B}_k} |B_{\mathfrak{b}}| \rightarrow |B|$ as $k \rightarrow \infty$, whence (29), in view of $\epsilon > 0$ and (27), shows that

$$\begin{aligned} |A + B| &\geq \left(1 + \frac{1}{\lambda}\right)^{d-1} |A| + (1 + \lambda)^{d-1} |B| + \frac{\delta^d (\lambda + 1)^{d-2}}{2\lambda^{d-1}} \epsilon \\ &> \left(1 + \frac{|T|^{1/(d-1)}}{|S|^{1/(d-1)}}\right)^{d-1} |A| + \left(1 + \frac{|S|^{1/(d-1)}}{|T|^{1/(d-1)}}\right)^{d-1} |B| \\ &= (|S|^{1/(d-1)} + |T|^{1/(d-1)})^{d-1} \left(\frac{|A|}{|S|} + \frac{|B|}{|T|} \right), \end{aligned}$$

contrary to hypothesis. □

We conclude the section with the proof of Theorem 1.1 for general convex bodies.

Proof of Theorem 1.1. As noted at the beginning of Section 2, we may w.l.o.g. assume π is the vertical projection map with $\mathbf{u} = \mathbf{e}_0$. Since a pair of homothetic convex bodies A' and B' attains equality in the Brunn-Minkowski inequality, and thus also in (6), Lemma 2.2 shows that the sets described by Theorem 1.1 all satisfy equality (6).

It remains to complete the other direction in Theorem 1.1, so assume $A, B \subseteq \mathbb{R}^d$ are convex bodies satisfying (6). Let $S = \pi(A)$ and $T = \pi(B)$, so that $M = |S|$ and $N = |T|$. In view of Lemma 2.1, it follows that S and T are homothetic convex bodies, say $S = \lambda T + \mathbf{x}_0$ with $\lambda > 0$ and $\mathbf{x}_0 \in \mathbb{R}^d$, and by translating appropriately, we may w.l.o.g. assume that $\mathbf{x}_0 = 0$. Write A and B using the notation of (5). Note that $v_A(\mathbf{x}) \geq u_A(\mathbf{x})$ and $v_B(\mathbf{y}) \geq u_B(\mathbf{y})$ for all $\mathbf{x} \in S$ and $\mathbf{y} \in T$. Let

$$\alpha = \inf_{\mathbf{x} \in S} \{v_A(\mathbf{x}) - u_A(\mathbf{x})\} \geq 0 \quad \text{and} \quad \beta = \inf_{\mathbf{x} \in T} \{v_B(\mathbf{x}) - u_B(\mathbf{x})\} \geq 0.$$

Since S and T are both compact subsets, these finite infima are attained by some $\mathbf{v} \in S$ and $\mathbf{v}' \in T$ (which, of course, may not be the only points for which the minimum is attained).

Let $A' \subseteq \mathbb{R}^d$ be the subset with $\pi(A') = \pi(A) = S$ defined by

$$A' = \{(y, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} \mid \mathbf{x} \in \pi(A), u_A(\mathbf{x}) \leq y \leq v_A(\mathbf{x}) - \alpha\}.$$

Then $u_{A'} = u_A$ and $v_{A'} = v_A - \alpha$ (in view of the definition of α) so that A' is the maximal vertical ‘compression’ of A . In particular, A is a vertical stretching of A' of amount α . Likewise, let $B' \subseteq \mathbb{R}^d$ be the subset with $\pi(B') = \pi(B) = T$ defined by

$$B' = \{(y, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} \mid \mathbf{x} \in \pi(B), u_B(\mathbf{x}) \leq y \leq v_B(\mathbf{x}) - \beta\}.$$

The set B is a vertical stretching of B' of amount β . In view of Lemma 2.2, we find that the pair A' and B' also satisfies Bonnesen’s equality (6).

Since $S = \lambda T$, it is easily observed that, if the graphs of v_A and v_B are both homothetic as well as the graphs of u_A and u_B , then we can take $\mathbf{v}' = \frac{1}{\lambda}\mathbf{v}$ and, moreover, A' and B' will then be homothetic convex bodies as the graphs of $y = v_A(x)$ and $y = u_A(x) - \alpha$ intersect over the point \mathbf{v} while the graphs of $y = v_B(x)$ and $y = u_B(x) - \beta$ intersect over the corresponding point $\mathbf{v}' = \frac{1}{\lambda}\mathbf{v}$, which would complete the proof in view of the comments of the previous paragraph. We proceed to show this is the case.

In view of S and T being homothetic and Lemma 2.2, we see that, to complete the proof, it suffices to prove the pair of graphs v_A and v_B and the pair of graphs u_A and u_B are both homothetic for any pair of vertical stretchings \tilde{A} and \tilde{B} of A and B . Thus, stretching A and B sufficiently, we may w.l.o.g. assume

$$\inf_{\mathbf{x} \in S} v_A(\mathbf{x}) > \sup_{\mathbf{x} \in S} u_A(\mathbf{x}) \quad \text{and} \quad \inf_{\mathbf{y} \in T} v_B(\mathbf{y}) > \sup_{\mathbf{y} \in T} u_B(\mathbf{y}).$$

Consequently, translating A and B appropriately, we can assume that $v_A(\mathbf{x}) > 0$ and $u_A(\mathbf{x}) < 0$ for all $\mathbf{x} \in S$, and that $v_B(\mathbf{y}) > 0$ and $u_B(\mathbf{y}) < 0$ for all $\mathbf{y} \in T$.

Let

$$\begin{aligned} A^+ &= A \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1}) \quad \text{and} \quad A^- = A \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^{d-1}), \\ B^+ &= B \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1}) \quad \text{and} \quad B^- = B \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^{d-1}). \end{aligned}$$

Then

$$\begin{aligned} A^+ &= -\text{epi}^*(-v_A) \quad \text{and} \quad B^+ = -\text{epi}^*(-v_B), \\ -A^- &= -\text{epi}^*(-(-u_A)) \quad \text{and} \quad -B^- = -\text{epi}^*(-(-u_B)). \end{aligned}$$

Since A and B are convex bodies, we have v_A and v_B being concave functions and u_A and u_B convex functions, in which case $-u_A$ and $-u_B$ are concave functions.

Since $A^+ + B^+ \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1}$ and $A^- + B^- \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^{d-1}$, we see that $(A^+ + B^+) \cap (A^- + B^-)$ is a measure zero subset. Thus, applying Theorem C to $A^+ + B^+$ and $A^- + B^-$,

it follows that

$$\begin{aligned}
 |A + B| &\geq |A^+ + B^+| + |A^- + B^-| \\
 &\geq (M^{1/(d-1)} + N^{1/(d-1)})^{d-1} \left(\frac{|A^+|}{M} + \frac{|B^+|}{N} \right) + \\
 &\quad (M^{1/(d-1)} + N^{1/(d-1)})^{d-1} \left(\frac{|A^-|}{M} + \frac{|B^-|}{N} \right) \\
 &= (M^{1/(d-1)} + N^{1/(d-1)})^{d-1} \left(\frac{|A|}{M} + \frac{|B|}{N} \right).
 \end{aligned}$$

By hypothesis, equality must hold in the above bound, which is only possible if equality held in both the estimates for $A^+ + B^+$ and for $A^- + B^-$. As result, applying Lemma 2.5 to $A^+ + B^+$ and to $(-A^-) + (-B^-)$ shows that the graphs of v_A and v_B are homothetic as well as the graphs of u_A and u_B , completing the proof. \square

3. EQUALITY IN THE HYPERPLANE SLICE BONNESEN BOUND FOR $d = 2$

In this section, we prove Theorem 1.2, thus determining the structure of extremal convex bodies satisfying Theorem B in dimension 2. To do so, by rotating appropriately, we can w.l.o.g. assume $H = \mathbf{e}_1^\perp$ is the \mathbf{e}_0 -axis. We begin with the following lemma, which does not necessarily hold for higher dimensions.

Lemma 3.1. *Let $A \subseteq \mathbb{R}^2$ be a convex body and let $H = \mathbb{R}\mathbf{e}_0$. Suppose A is translated so that*

$$|H \cap A| = \sup\{|(\mathbf{x} + H) \cap A| \mid \mathbf{x} \in \mathbb{R}^2\}.$$

Then there exists some linear transformation $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\pi(\mathbb{R}^2) = H = \mathbb{R}\mathbf{e}_0$ and $\pi(A) = H \cap A$.

Proof. Let $H \cap A = [m, n] \times \{0\}$ with $m < n$. Write A using the notation of (5) (taking $\mathbf{u} = \mathbf{e}_0$) and simplifying the notation for u_A and v_A by defining $u := u_A$ and $v := v_A$. Observe that $u(0) = m$ and $v(0) = n$. To prove the lemma, we need to find a slope λ so that the line passing through $(m, 0)$ with slope λ as well as the line passing through $(n, 0)$ with slope λ are both supporting/tangent lines to A , as then the linear transformation $\pi : \mathbb{R}^2 \rightarrow H$ having the line of slope λ as its kernel will satisfy the conclusions of the lemma. However, in terms of subdifferentials, this is equivalent to showing $\partial u(0) \cap -\partial(-v)(0)$ is nonempty.

Define $\tilde{f} : \pi(A) \rightarrow \mathbb{R}$ by $\tilde{f}(x) = u(x) - v(x)$. Note $-\tilde{f}(x) = |((0, x) + H) \cap A|$, so that, by hypothesis, $\min \tilde{f} = \tilde{f}(0) = m - n < 0$. As A is convex, we know $-v$ and u are both convex functions. Hence, since the sum of convex functions remains convex, we see that \tilde{f} is a convex function. Since $\tilde{f}(x)$ attains its minimum at $x = 0$, we must have $0 \in \partial \tilde{f}(0)$,

which means

$$(30) \quad \tilde{f}'_-(0) \leq 0 \leq \tilde{f}'_+(0).$$

From the definition of the one-sided derivative, it follows that

$$(31) \quad \tilde{f}'_+ = u'_+ + (-v)'_+ \quad \text{and} \quad \tilde{f}'_- = u'_- + (-v)'_-.$$

Suppose by contradiction that $\partial u(0) \cap -\partial(-v)(0) = \emptyset$. Then, since

$$\partial u(0) = [u'_-(0), u'_+(0)] \quad \text{and} \quad -\partial(-v)(0) = [-(v)'_+(0), -(v)'_-(0)],$$

we see that either

$$u'_+(0) < -(v)'_+(0) \quad \text{or} \quad -(v)'_-(0) < u'_-(0).$$

Consequently, it follows in view of (31) that either

$$\tilde{f}'_+(0) = u'_+(0) + (-v)'_+(0) < 0 \quad \text{or} \quad \tilde{f}'_-(0) = u'_-(0) + (-v)'_-(0) > 0,$$

contradicting (30). \square

We now proceed with the simple derivation of Theorem 1.2 from Theorem 1.1.

Proof of Theorem 1.2. If A and B are a pair of sets satisfying the description given by Theorem 1.2, then the equality (7) is the same as the equality (6), which holds for A and B in view of Theorem 1.1.

It remains to complete the other direction of Theorem 1.2, so assume $A, B \subseteq \mathbb{R}^2$ are convex bodies satisfying (7). By rotating appropriately, we can assume w.l.o.g. that $H = \mathbf{e}_1^\perp = \mathbb{R}\mathbf{e}_0$ is the \mathbf{e}_0 -axis. We may also w.l.o.g. assume

$$(32) \quad \frac{|B|}{N^2} \leq \frac{|A|}{M^2}.$$

In view of Lemma 3.1, let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation such that $\pi(\mathbb{R}^2) = H = \mathbb{R}\mathbf{e}_0$ and $\pi(A) = H \cap A$. Let $N' = |\pi(B)|$. Note, since $H \cap B \subseteq \pi(B)$ and $N = |H \cap B|$, that

$$N \leq N'.$$

Applying Theorem C using π , we conclude that

$$(33) \quad |A + B| \geq (M + N') \left(\frac{|A|}{M} + \frac{|B|}{N'} \right).$$

Let $h : \mathbb{R}_{>0}^2 \rightarrow \mathbb{R}$ be defined by $h(x, y) = (x + y) \left(\frac{|A|}{x} + \frac{|B|}{y} \right)$. Then

$$h(M, N) = (M + N) \left(\frac{|A|}{M} + \frac{|B|}{N} \right) = |A + B|$$

by hypothesis. Letting $h_x(y) = h(x, y)$, we find that $h'_x(y) = \frac{-|B|x}{y^2} + \frac{|A|}{x}$, which is non-negative when

$$(34) \quad \frac{|B|}{y^2} \leq \frac{|A|}{x^2},$$

and positive when $\frac{|B|}{y^2} < \frac{|A|}{x^2}$.

If $(x, y) \in \mathbb{R}^2$ satisfies (34) with $x, y > 0$, then (x, y') will satisfy (34) strictly for all $y' > y$. Consequently, it follows from the above derivative analysis that $h(x, y') > h(x, y)$ for such (x, y) . In particular, in view of (32) and $N' \geq N$, we see that $h(M, N') \geq h(M, N)$ with equality possible only if $N' = N$. As a result, since $|A + B| = h(M, N)$ holds with equality by hypothesis, we conclude from (33) that $N = N'$. Therefore, since $H \cap B \subseteq \pi(B)$ with $|H \cap B| = N = N' = |\pi(B)|$, we see that $\pi(B) \setminus (H \cap B)$ is a measure zero subset. Thus, since $B \subseteq \mathbb{R}^2$ is a convex body, so that $\pi(B)$ and $H \cap B$ are both closed intervals in \mathbb{R} , it follows that $\pi(B) = H \cap B$. Hence, since we also have $\pi(A) = H \cap A$ by the choice of π , we see that applying Theorem 1.1 with π completes the proof. \square

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