ZERO-SUM PROBLEMS WITH CONGRUENCE CONDITIONS

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ABSTRACT. For a finite abelian group G and a positive integer d, let $\mathsf{s}_{d\mathbb{N}}(G)$ denote the smallest integer $\ell \in \mathbb{N}_0$ such that every sequence S over G of length $|S| \ge \ell$ has a nonempty zero-sum subsequence T of length $|T| \equiv 0 \mod d$. We determine $\mathsf{s}_{d\mathbb{N}}(G)$ for all $d \ge 1$ when G has rank at most two and, under mild conditions on d, also obtain precise values in the case of p-groups. In the same spirit, we obtain new upper bounds for the Erdős–Ginzburg–Ziv constant provided that, for the p-subgroups G_p of G, the Davenport constant $\mathsf{D}(G_p)$ is bounded above by $2\exp(G_p) - 1$. This generalizes former results for groups of rank two.

1. Introduction

Let G be an additive finite abelian group. A direct zero-sum problem, associated to a given Property P, asks for the extremal conditions which guarantee that every sequence S over G satisfying these conditions has a zero-sum subsequence with Property P. Most of the properties studied so far deal with the length of the zero-sum subsequence; others consider the cross number (see, e.g., [14]) or versions of this problem involving weights (see, e.g., [1]). In the case of lengths, a direct zero-sum problem asks for the smallest integer $\ell \in \mathbb{N}_0$ such that every sequence S over G of length $|S| \ge \ell$ has a zero-sum subsequence of some prescribed length. This leads to the definition of the following zero-sum constant:

For a subset $L \subset \mathbb{N}$, let $\mathsf{s}_L(G)$ denote the smallest $\ell \in \mathbb{N}_0 \cup \{\infty\}$ such that every sequence S over G of length $|S| \ge \ell$ has a zero-sum subsequence T of length $|T| \in L$.

Note that $\mathsf{s}_L(G) = \infty$ if and only if $L \cap \exp(G)\mathbb{N} = \emptyset$. The following sets lead to classical zero-sum invariants (the reader may want to consult one of the surveys [8, 15] or the monograph [18]):

- $\mathsf{s}_{\mathbb{N}}(G) = \mathsf{D}(G)$ is the Davenport constant,
- $\mathsf{s}_{\{\exp(G)\}}(G) = \mathsf{s}(G)$ is the Erdős–Ginzburg–Ziv constant,
- $\mathsf{s}_{\{|G|\}}(G) = \mathsf{ZS}(G)$ is the zero-sum constant, and
- $\mathsf{s}_{[1,\exp(G)]}(G) = \eta(G)$ is the η-invariant.

Moreover, $\mathsf{s}_L(G)$ has been investigated for various other sets, including: $[1, k]$ for $k \ge \exp(G)$ (see, e.g., [4, 2, 6]), $\{k\exp(G)\}$ for $k \in \mathbb{N}$ (see, e.g., [13, 26]), $\mathbb{N} \setminus k\mathbb{N}$ for $k \nmid \exp(G)$ and other unions of arithmetic progressions (see [7, 29, 21]), and $\exp(G)\mathbb{N}$ (see, e.g., [3]). And, for recent closely related results, see, e.g., [22, 9, 12, 23, 11, 31].

In the present paper, we investigate $\mathsf{s}_{d\mathbb{N}}(G)$, first proving upper and lower bounds in terms of a Davenport constant and its canonical lower bound. This allows us to determine $\mathsf{s}_{d\mathbb{N}}(G)$ for cyclic groups and, under mild conditions on d, for p-groups (Theorem 3.1). Then we suppose that $d = \exp(G)$ and that, for the p-subgroups G_p of G, the Davenport constant $\mathsf{D}(G_p)$ is bounded above by $2\exp(G_p) - 1$ (note that every group of rank at most two satisfies this condition). In this setting, we obtain canonical upper bounds for $\mathsf{s}_{d\mathbb{N}}(G)$ and, among others, for the Erdős–Ginzburg–Ziv constant $\mathsf{s}(G)$ (Theorem 4.1, and see Theorem 4.2 for a result in a similar vein). Next, using a more involved argument, we determine $\mathsf{s}_{d\mathbb{N}}(G)$

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for rank 2 groups G, showing that $\mathsf{s}_{d\mathbb{N}}(G)$ attains the value that would easily follow from our bounds if the conjectured value of $\mathsf{D}(G)$ for rank 3 groups were true. In the final section, we apply these results to a problem from the theory of non-unique factorizations which motivated the present investigations.

Throughout this paper, let G be a finite abelian group.

2. Preliminaries

Our notation and terminology are consistent with [10] and [18]. We briefly gather some key notions and fix the notation concerning sequences over abelian groups. Let $\mathbb{N}$ denote the set of positive integers, let $\mathbb{P} \subset \mathbb{N}$ be the set of prime numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. For $n \in \mathbb{N}$ and $p \in \mathbb{P}$, let C_n denote a cyclic group with n elements and $\mathsf{v}_p(n) \in \mathbb{N}_0$ the p-adic valuation of n with $\mathsf{v}_p(p) = 1$. Throughout, all abelian groups will be written additively.

For a subset $G_0 \subset G$, let $\langle G_0 \rangle$ denote the subgroup generated by G_0. For a prime $p \in \mathbb{P}$, we denote by $G_p = \{g \in G \mid \mathrm{ord}(g) \text{ is a power of } p\}$ the p-primary component of G. Suppose that $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $r \in \mathbb{N}_0$ and $1 < n_1 \mid \ldots \mid n_r$. Then $r = \mathsf{r}(G)$ will be called the *rank* of G, and we set

$$\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r} (n_i - 1)\,.$$

Note that $\mathsf{r}(G) = 0$ and $\mathsf{D}^*(G) = 1$ for G trivial.

Let $\mathcal{F}(G)$ be the free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}\,, \quad \text{with} \quad \mathsf{v}_g(S) \in \mathbb{N}_0 \quad \text{for all} \quad g \in G\,.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and we say that S *contains* g if $\mathsf{v}_g(S) > 0$. A sequence S_1 is called a *subsequence* of S if $S_1 \,|\, S$ in $\mathcal{F}(G)$ (equivalently, $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$). If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence

$$S \;=\; g_1 \cdot \ldots \cdot g_l \;=\; \prod_{g \in G} g^{\mathsf{v}_g(S)} \;\in \mathcal{F}(G)\,,$$

we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \qquad \text{the } \textit{length} \text{ of } S\,,$$

$$\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \qquad \text{the } \textit{support} \text{ of } S \quad \text{and}$$

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \qquad \text{the } \textit{sum} \text{ of } S\,.$$

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if there is no nonempty zero-sum subsequence,
- a *minimal zero-sum sequence* if S is a nonempty zero-sum sequence and every $S' | S$ with $1 \le |S'| < |S|$ is zero-sum free.

Every map of abelian groups $\varphi \colon G \to H$ extends to a homomorphism $\varphi \colon \mathcal{F}(G) \to \mathcal{F}(H)$ where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. If φ is a homomorphism, then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \mathrm{Ker}(\varphi)$. We let $\mathcal{A}(G)$ denote the set of all minimal zero-sum sequences over G.

3. Basic bounds and results for cyclic and p-groups

In this section, we establish some of our results on $\mathsf{s}_{d\mathbb{N}}(G)$. In particular, we obtain the following result.

Theorem 3.1. *Let $d \in \mathbb{N}$ and let $n = \exp(G)$.*

1. *Suppose G is cyclic. Then $\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d) = \mathrm{lcm}(n, d) + \gcd(n, d) - 1$.*
2. *Suppose G is a p-group.*
   (a) *$\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d) = \mathsf{D}^*(G) + d - 1$ for $d = p^\alpha$ with $\alpha \in \mathbb{N}_0$.*
   (b) *$\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d) = \mathsf{D}^*(G) + d - 1$ for each $d \in \mathbb{N}$ with $\mathsf{D}^*(G) \le p^{\mathsf{v}_p(d)}$.*
   (c) *$\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d) = \mathsf{D}^*(G) - n + \mathrm{lcm}(n, d) + \gcd(n, d) - 1$ for each $d \in \mathbb{N}$ with $p^{\mathsf{v}_p(d)} \le 2n - \mathsf{D}^*(G)$.*

The strategy to prove this result is to bound $\mathsf{s}_{d\mathbb{N}}(G)$, for generic d and G, in terms of the invariants $\mathsf{D}^*(\cdot)$ and $\mathsf{D}(\cdot)$, and then to make these 'abstract bounds' explicit invoking results on the Davenport constant. We remark that Theorem 3.1.2(a) for $\alpha = 1$ can also be derived as a special case of [21, Theorem 2.3], proved via the Combinatorial Nullstellensatz. The first part is carried out in Proposition 3.3. However, since the lower bound given there is in terms of $\mathsf{D}^*(G \oplus C_d)$, we begin first with the following lemma showing how to calculate $\mathsf{D}^*(G \oplus C_d)$ explicitly.

Lemma 3.2. *Let $d \in \mathbb{N}$ and let $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. Set $n_0 = 1$ and $n_{r+1} = 0$. Then $G \oplus C_d \cong C_{m_0} \oplus \cdots \oplus C_{m_r}$ with $1 \le m_0 \mid \cdots \mid m_r$, so that*

$$\mathsf{D}^*(G \oplus C_d) = \sum_{i=0}^{r} (m_i - 1) + 1,$$

where

$$m_i = n_i \frac{\gcd(n_{i+1}, d)}{\gcd(n_i, d)} = \gcd(n_{i+1}, \mathrm{lcm}(n_i, d)) = \mathrm{lcm}(n_i, \gcd(n_{i+1}, d)) \quad \text{for } i \in [0, r].$$

Proof. To prove this result, we first use the composition of the groups in cyclic components of prime power order, where the relation between the compositions of G and $G \oplus C_d$ is transparent, and reconstruct the desired description of $G \oplus C_d$. Due to its central nature for the remainder of the paper, we provide full details of the routine argument. Letting $p_1, \ldots, p_k$ be the distinct prime divisors p_i of n_r, we have

$$G \cong \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{r} C_{p_i^{s_{i,j}}},$$

where $0 \le s_{i,1} \le \ldots \le s_{i,r}$ for all $i \in [1, k]$ and $p_1^{s_{1,j}} \cdot \ldots \cdot p_k^{s_{k,j}} = n_j$ for all $j \in [1, r]$. Let

$$C_d \cong C_m \oplus \bigoplus_{i=1}^{k} C_{p_i^{s'_i}},$$

where $\mathsf{v}_{p_i}(d) = s'_i \ge 0$ and $d = m p_1^{s'_1} \cdots p_k^{s'_k}$. Recall $n_0 = 1$, $n_{r+1} = 0$ and write

$$G \oplus C_d \cong C_{m_0} \oplus \ldots \oplus C_{m_r} \tag{3.1}$$

with $1 \le m_0 \mid \cdots \mid m_r$. Then

$$\mathsf{v}_{p_i}(m_j) = \begin{cases} \mathsf{v}_{p_i}(n_j) = s_{i,j}, & \text{if } \mathsf{v}_{p_i}(d) \le \mathsf{v}_{p_i}(n_j), \\ \mathsf{v}_{p_i}(d) = s'_i, & \text{if } \mathsf{v}_{p_i}(n_j) \le \mathsf{v}_{p_i}(d) \le \mathsf{v}_{p_i}(n_{j+1}), \\ \mathsf{v}_{p_i}(n_{j+1}) = s_{i,j+1}, & \text{if } \mathsf{v}_{p_i}(n_{j+1}) \le \mathsf{v}_{p_i}(d), \end{cases} \tag{3.2}$$

for $j \in [0, r]$ and $i \in [1, k]$.

We claim that

$$m_j = n_j \frac{\gcd(n_{j+1}, d)}{\gcd(n_j, d)}, \tag{3.3}$$

for $j \in [0, r]$. Indeed, since $m_r = \exp(G \oplus C_d) = n_r \frac{d}{\gcd(n_r, d)}$, this is clear for $j = r$. To see this also holds for $j < r$, it suffices to see $\mathsf{v}_{p_i}\left(n_j \frac{\gcd(n_{j+1},d)}{\gcd(n_j,d)}\right)$ agrees with (3.2) for each $i \in [1, k]$. However,

$$\mathsf{v}_{p_i}\left(n_j \frac{\gcd(n_{j+1},d)}{\gcd(n_j,d)}\right) = \mathsf{v}_{p_i}(n_j) + \min\{\mathsf{v}_{p_i}(d),\, \mathsf{v}_{p_i}(n_{j+1})\} - \min\{\mathsf{v}_{p_i}(d),\, \mathsf{v}_{p_i}(n_j)\},$$

which is easily seen to agree with (3.2) in all three cases, completing the claim. Thus from (3.1) we conclude that

$$\mathsf{D}^*(G \oplus C_d) = \sum_{i=0}^{r} (m_i - 1) + 1.$$

Moreover, in view of (3.3), one sees that the expression for the m_i can be rewritten as

$$m_i = \gcd(n_{i+1}, \operatorname{lcm}(n_i, d)) = \operatorname{lcm}(n_i, \gcd(n_{i+1}, d)) \quad \text{for } i \in [0, r],$$

which completes the proof. □

Now, we establish our 'abstract bounds.' A main point is the inequality $\mathsf{D}^*(G \oplus C_d) \le \mathsf{s}_{d\mathbb{N}}(G)$, which can be a considerably better bound than the more direct lower bound of $\mathsf{D}(G) + d - 1$.

Proposition 3.3. *Let $d \in \mathbb{N}$. Then*

$$\mathsf{D}^*(G) + d - 1 \le \mathsf{D}^*(G \oplus C_d) \le \mathsf{s}_{d\mathbb{N}}(G) \le \mathsf{D}(G \oplus C_d)$$

and

$$\mathsf{D}(G) + d - 1 \le \mathsf{s}_{d\mathbb{N}}(G).$$

In particular, if $\mathsf{D}^*(G \oplus C_d) = \mathsf{D}(G \oplus C_d)$, then $\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d)$.

Proof. Write $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$, where $1 < n_1 \mid \cdots \mid n_r$, let $n_0 = 1$ and $n_{r+1} = 0$, and let $e_1, \ldots, e_r \in G$ be such that $G = \oplus_{i=1}^r \langle e_i \rangle$ with $\operatorname{ord}(e_i) = n_i$. We begin by showing $\mathsf{D}^*(G) + d - 1 \le \mathsf{D}^*(G \oplus C_d)$, which is a fairly direct consequence of Lemma 3.2. By Lemma 3.2, we know

$$\mathsf{D}^*(G \oplus C_d) = \sum_{i=0}^{r} (m_i - 1) + 1, \tag{3.4}$$

where

$$m_i = n_i \frac{\gcd(n_{i+1}, d)}{\gcd(n_i, d)} = \gcd(n_{i+1}, \operatorname{lcm}(n_i, d)) = \operatorname{lcm}(n_i, \gcd(n_{i+1}, d)) \quad \text{for } i \in [0, r].$$

From (3.4), we have

$$\mathsf{D}^*(G \oplus C_d) = \sum_{i=0}^{r} m_i - r = \sum_{i=0}^{r} d_i n_i - r,$$

where

$$d_i = \frac{\gcd(n_{i+1}, d)}{\gcd(n_i, d)} \quad \text{for } i \in [0, r]. \tag{3.5}$$

Observing that $d_0 \cdots d_r = d$ with $d_i \in [1, d]$ for all i, and noting that $d_0 n_0 = d_0 = \gcd(n_1, d) \mid n_j$ for all j, so that $d_0 n_0 \le n_j$, it is easily verified that the above expression is minimized when $d_0 = d$ and $d_i = 1$ for $i \ge 1$, in which case $\mathsf{D}^*(G \oplus C_d) \ge d + \sum_{i=1}^{r} n_i - r = \mathsf{D}^*(G) + d - 1$, as desired.

Next, we show $\mathsf{D}(G) + d - 1 \le \mathsf{s}_{d\mathbb{N}}(G)$. By definition of $\mathsf{D}(G)$, there exists a zero-sum free sequence $S \in \mathcal{F}(G)$ with $|S| = \mathsf{D}(G) - 1$. We consider the sequence $0^{d-1}S$. Clearly, the only nonempty zero-sum subsequences of $0^{d-1}S$ are the sequences 0^k with $k \in [1, d-1]$. Thus $\mathsf{s}_{d\mathbb{N}}(G) > |0^{d-1}S| = \mathsf{D}(G) + d - 2$, establishing our claim.

We proceed to show the remaining lower bound $\mathsf{D}^*(G \oplus C_d) \le \mathsf{s}_{d\mathbb{N}}(G)$. We give an example of a sequence of length $\mathsf{D}^*(G \oplus C_d) - 1$ without a zero-sum subsequence of the desired length. This example is suggested by the description of $G \oplus C_d$ as given in Lemma 3.2. Let $e_0 = 0$ and let $S = \prod_{i=0}^{r} e_i^{m_i - 1}$. From (3.4), we have $|S| = \mathsf{D}^*(G \oplus C_d) - 1$. Consider $T \mid S$ with $\sigma(T) = 0$ and $d \mid |T|$. We will show that $|T| = 0$, establishing the lower bound. Let $v_i = \mathsf{v}_{e_i}(T)$ for $i \in [0, r]$. We note that $n_i = \operatorname{ord}(e_i) \mid v_i$ for each i, and we set $x_i = v_i/n_i$. By the very definition, we have

$$|T| = \sum_{i=0}^{r} x_i n_i.$$

Note that $v_i = x_i n_i < m_i$ (as $\mathsf{v}_{e_i}(S) < m_i$ with $T \mid S$), and thus $x_i \in [0, d_i - 1]$ for each i. We have to show that $x_i = 0$ for each i. Assume not, and let $j \in [0, r]$ be minimal with $x_j \neq 0$. Since

$$|T| = \sum_{i=1}^{r} x_i n_i = \sum_{i=j}^{r} x_i n_i$$

is divisible by d, we get that (for $j = r$, the right-hand side below is 0)

$$x_j n_j \equiv -n_{j+1} \sum_{i=j+1}^{r} x_i \frac{n_i}{n_{j+1}} \pmod{d},$$

and thus $\gcd(n_{j+1}, d) \mid x_j n_j$. Consequently, $\frac{\gcd(n_{j+1}, d)}{\gcd(n_j, d)} \mid \frac{x_j n_j}{\gcd(n_j, d)}$, whence (3.5) implies

$$d_j \mid x_j \frac{n_j}{\gcd(n_j, d)}.$$

Noting from (3.5) that

$$\gcd(d_j, \frac{n_j}{\gcd(n_j, d)}) = 1,$$

it follows that $d_j \mid x_j$, which in view of $x_j \in [0, d_j - 1]$ implies $x_j = 0$. This contradicts the definition of x_j and completes the argument.

It remains to show $\mathsf{s}_{d\mathbb{N}}(G) \leq \mathsf{D}(G \oplus C_d)$, which we achieve by a standard imbedding argument. Let $S \in \mathcal{F}(G)$ with $|S| \geq \mathsf{D}(G \oplus C_d)$. We have to show that S has a nonempty zero-sum subsequence of length congruent to 0 modulo d. Let $e \in G \oplus C_d$ be such that $G \oplus C_d = G \oplus \langle e \rangle$, and let $\iota \colon G \to G \oplus C_d$ denote the map defined via $\iota(g) = g + e$. Since $|\iota(S)| = |S| \geq \mathsf{D}(G \oplus C_d)$, applying the definition of $\mathsf{D}(G \oplus C_d)$ to $\iota(S)$ yields a nonempty subsequence $T \mid S$ with $0 = \sigma(\iota(T)) = \sigma(T) + |T|e$. Hence T is a zero-sum subsequence with length $|T|$ divisible by $\operatorname{ord}(e) = n$, as desired. □

Now we prove Theorem 3.1. We need the following well-known results on the Davenport constant, which will be used later in the paper as well. Namely, $\mathsf{D}(G) = \mathsf{D}^*(G)$ if G satisfies any one of the following conditions (see [15], specifically Theorems 2.2.6 and 4.2.10 and Corollary 4.2.13):

- G has rank at most two.
- G is a p-group.
- $G \cong G' \oplus C_n$ where G' is a p-group with $\mathsf{D}^*(G') \leq 2\exp(G') - 1$ and $p \nmid n$.

Proof of Theorem 3.1. The proof is a combination of the bounds obtained in Proposition 3.3 and known results for the Davenport constant.

1. As G is cyclic, it follows from Lemma 3.2 that $G \oplus C_d \cong C_{\gcd(n,d)} \oplus C_{\operatorname{lcm}(n,d)}$. Thus, by the above mentioned results, we know that $\mathsf{D}(C_{\gcd(n,d)} \oplus C_{\operatorname{lcm}(n,d)}) = \mathsf{D}^*(C_{\gcd(n,d)} \oplus C_{\operatorname{lcm}(n,d)}) = \gcd(n, d) + \operatorname{lcm}(n, d) - 1$, whence Proposition 3.3 completes the proof of part 1.

2. Let H be a group such that $G \cong H \oplus C_n$. As G is a p-group, it follows from Lemma 3.2 that

$$G \oplus C_d \cong H \oplus C_{\gcd(n,d)} \oplus C_{\operatorname{lcm}(n,d)}$$

with $\operatorname{lcm}(n, d)$ the exponent of $G \oplus C_d$. Consequently, since G is a p-group, it follows that $\mathsf{D}^*(G \oplus C_d) = \mathsf{D}^*(H) + \gcd(n, d) + \operatorname{lcm}(n, d) - 2$. Observe that this quantity is equal to the value we claim for $\mathsf{s}_{d\mathbb{N}}(G)$ in each of the points (a), (b), and (c), with this being the case in (b) since $p^{\mathsf{v}_p(d)} \geq \mathsf{D}^*(G) \geq n$ with n being a power of p (as G is a p-group) implies $\mathrm{lcm}(n,d) = d$ and $\gcd(n,d) = n$. Thus, again, by Proposition 3.3 it suffices to show that $\mathsf{D}(G \oplus C_d) = \mathsf{D}^*(G \oplus C_d)$. For (a), $G \oplus C_d$ is a p-group and the claim is immediate by the above mentioned result for p-groups. For (b) and (c), let $\alpha_1 \in \mathbb{N}_0$ be such that $\gcd(n,d) = p^{\alpha_1}$ and let $\alpha_2 = \mathsf{v}_p(\mathrm{lcm}(n,d))$.

Suppose the hypotheses of (b) hold. Then $n \leq \mathsf{D}^*(G) \leq p^{\mathsf{v}_p(d)}$ so that $p^{\alpha_2} = p^{\mathsf{v}_p(d)}$ and $p^{\alpha_1} = n$. Hence, using the hypothesis $n \leq \mathsf{D}^*(G) \leq p^{\mathsf{v}_p(d)} = p^{\alpha_2}$ once more, we find that

$$\begin{aligned} \mathsf{D}(H \oplus C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}}) &= \mathsf{D}^*(H \oplus C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}}) \\ &= \mathsf{D}^*(G) + p^{\alpha_2} - 1 \leq p^{\mathsf{v}_p(d)} + p^{\alpha_2} - 1 = 2p^{\alpha_2} - 1. \end{aligned}$$

Thus the p-group $H \oplus C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}}$ fulfils the conditions imposed in the last of the above mentioned results, completing the proof of (b).

Suppose the hypotheses of (c) hold. If $p^{\alpha_2} = p^{\mathsf{v}_p(d)}$, then $n \leq p^{\mathsf{v}_p(d)}$, whence the hypothesis of (c) implies $\mathsf{D}^*(G) \leq n$. As a result, since $n \leq \mathsf{D}^*(G)$ with equality if and only if G is cyclic, we conclude that G is cyclic. Consequently, $G \oplus C_d$ has rank at most 2, so that $\mathsf{D}^*(G \oplus C_d) = \mathsf{D}(G \oplus C_d)$ by the first of the above mentioned results, and now the result follows from Proposition 3.3. Therefore it remains to consider the case when $p^{\alpha_2} = n$ and $p^{\alpha_1} = p^{\mathsf{v}_p(d)}$. In this case, the hypothesis of (c) instead implies

$$\begin{aligned} \mathsf{D}(H \oplus C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}}) &= \mathsf{D}^*(H \oplus C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}}) \\ &= \mathsf{D}^*(G) + p^{\alpha_1} - 1 \leq 2n - 1 = 2p^{\alpha_2} - 1. \end{aligned}$$

Thus the p-group $H \oplus C_{p^{\alpha_1}} \oplus C_{p^{\alpha_2}}$ fulfils the conditions imposed in the last of the above mentioned results, completing the proof of (c). $\square$

Several results on the Davenport constant, in addition to those already recalled, are known (see, e.g., [8] for an overview). Essentially, each of them allows one to obtain some additional insight on $\mathsf{s}_{d\mathbb{N}}(G)$ via Proposition 3.3. For example, it is conjectured that $\mathsf{D}^*(G) = \mathsf{D}(G)$ for groups of rank three (see [8, Conjecture 3.5]; and [2] and [28] for recent results, confirming this conjecture in special cases). If this were the case, then, for groups of rank two, $\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d)$ would immediately follow from Proposition 3.3 for all $d \in \mathbb{N}$. In Section 5, we will show this equality holds without the use of the conjectured value of $\mathsf{D}(G)$ for rank three groups, which could be construed as giving weak evidence for the supposed value.

Of course, the two invariants $\mathsf{D}(\cdot)$ and $\mathsf{D}^*(\cdot)$ are not equal for all finite abelian groups, and there are examples of pairs (d, G) for which the bounds in Proposition 3.3 do not coincide, i.e.,

$$\max\{\mathsf{D}(G) + d - 1, \mathsf{D}^*(G \oplus C_d)\} < \mathsf{D}(G \oplus C_d);$$

see [20, 19] for more information on the phenomenon of inequality of $\mathsf{D}(\cdot)$ and $\mathsf{D}^*(\cdot)$. However, it is conjectured that the difference between $\mathsf{D}(G)$ and $\mathsf{D}^*(G)$ is fairly small for any G (in a relative sense)—indeed, there is a conjecture that asserts that this difference is at most $\mathsf{r}(G) - 1$ (see [8, Conjecture 3.7])—and thus the combination of the bounds of Proposition 3.3 would in general yield a good approximation for $\mathsf{s}_{d\mathbb{N}}(G)$.

4. Results when $\mathsf{D}(G_p) \leq 2\exp(G_p) - 1$

We use the inductive method to obtain upper bounds on $\mathsf{D}(G)$, $\mathsf{s}(G)$, $\eta(G)$ and $\mathsf{s}_{d\mathbb{N}}(G)$, imposing conditions on the p-subgroups of G. These conditions are fulfilled, in particular, for groups of rank at most two. Recall that the question of whether or not $\mathsf{s}(C_p \oplus C_p) \leq 4p - 3$ holds for all primes $p \in \mathbb{P}$ was open for more then 20 years (the Kemnitz Conjecture), and finally solved by C. Reiher [27]. His result was then generalized to arbitrary groups of rank two [18, Theorem 5.8.3], and to p-groups G with $\mathsf{D}(G) \leq 2\exp(G) - 1$ [30, Theorem 1.2]. We refer to [15, Section 4] for a survey on the Erdős–Ginzburg–Ziv constant, and to [25, 24] for some recent connections. The upper bound for $\mathsf{s}_{n\mathbb{N}}(G)$ for groups of rank two was first given in [8, Theorem 6.7]. Note that the upper bound $\eta(G) \le 3n-2$ is precisely what is needed in various applications (see for example [18]).

Theorem 4.1. *Let* $\exp(G) = n$. *Suppose that, for each* $p \in \mathbb{P}$, *we have* $\mathsf{D}(G_p) \le 2\exp(G_p) - 1$.

1. *The following inequalities hold*:
   (a) $\mathsf{D}(G) \le 2n - 1$.
   (b) $\mathsf{s}_{n\mathbb{N}}(G) \le 3n - 2$.
2. *If* $\exp(G)$ *is odd, then the following inequalities hold*:
   (a) $\eta(G) \le 3n - 2$.
   (b) $\mathsf{s}(G) \le 4n - 3$.

In some cases, we are even able to establish the exact value of these constants, though we have to impose more restrictive conditions. We do not include $\mathsf{D}(G)$ in the result below since, in this case, an assertion of this form is well-known (see the result mentioned before the proof of Theorem 3.1). The relevance of these assumptions is discussed in some detail in Remark 4.3.

Theorem 4.2. *Let* $\exp(G) = n$. *Suppose there exists some odd* $q \in \mathbb{P}$ *such that* $\mathsf{D}(G_q) - \exp(G_q) + 1 \mid \exp(G_q)$ *and* G_p *is cyclic for each* $p \in \mathbb{P} \setminus \{q\}$.

1. $\eta(G) = 2(\mathsf{D}(G_q) - \exp(G_q)) + n$.
2. $\mathsf{s}(G) = 2(\mathsf{D}(G_q) - \exp(G_q)) + 2n - 1$.
3. $\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}(G_q) - \exp(G_q) + \gcd(n, d) + \operatorname{lcm}(n, d) - 1$ *for each* $d \in \mathbb{N}$ *with* $(\mathsf{D}(G_q) - \exp(G_q) + 1) \mid d$.

For both of the proofs below, we use [18, Proposition 5.7.11], which states that if $K \le G$ and $\exp(G) = \exp(K)\exp(G/K)$, then

$$\eta(G) \le \exp(G/K)(\eta(K) - 1) + \eta(G/K) \quad \text{and} \tag{4.1}$$

$$\mathsf{s}(G) \le \exp(G/K)(\mathsf{s}(K) - 1) + \mathsf{s}(G/K); \tag{4.2}$$

these inequalities are also obtained using the inductive method.

Proof of Theorem 4.1. Let $p_1, \ldots, p_s$ be the distinct primes such that $G = G_{p_1} \oplus \cdots \oplus G_{p_s}$ is the decomposition of G into non-trivial p-groups.

First, we establish the claims on $\eta(G)$ and $\mathsf{s}(G)$. Thus, we (temporarily) assume that each p_i is odd. We induct on s. For $s = 0$, the claim is trivial, and for $s = 1$, it is an immediate consequence of [30, Theorem 1.2], which asserts that, for H a p-group with p an odd prime and $\mathsf{D}(H) \le 2\exp(H) - 1$, one has $\eta(H) \le \mathsf{D}(H) + \exp(H) - 1$ and $\mathsf{s}(H) \le \mathsf{D}(H) + 2\exp(H) - 2$.

Suppose $s \ge 2$ and the claims hold true for $s-1$. Since $\exp(G) = \exp(G/G_{p_s})\exp(G_{p_s})$, we can invoke (4.1) and (4.2) to conclude

$$\eta(G) \le \exp(G/G_{p_s})(\eta(G_{p_s}) - 1) + \eta(G/G_{p_s}) \quad \text{and} \tag{4.3}$$

$$\mathsf{s}(G) \le \exp(G/G_{p_s})(\mathsf{s}(G_{p_s}) - 1) + \mathsf{s}(G/G_{p_s}). \tag{4.4}$$

By induction hypothesis, we have

$$\eta(G/G_{p_s}) \le 3\exp(G)/\exp(G_{p_s}) - 2, \quad \eta(G_{p_s}) \le 3\exp(G_{p_s}) - 2,$$
$$\mathsf{s}(G/G_{p_s}) \le 4\exp(G)/\exp(G_{p_s}) - 3, \ \text{and} \quad \mathsf{s}(G_{p_s}) \le 4\exp(G_{p_s}) - 3.$$

Combining these inequalities with (4.3) and (4.4) yields the desired bounds.

Next, we prove the result on $\mathsf{s}_{n\mathbb{N}}(G)$ and $\mathsf{D}(G)$. However, the upper bound on $\mathsf{D}(G)$ follows from Proposition 3.3 and part 1(b), so it suffices to show 1(b). To do so, we drop the assumption that each p_i is odd. Of course, at most one of the p_i's is even, and thus we may assume that $p_1, \ldots, p_{s-1}$ are odd. Again, we induct on s. The case $s = 0$ is trivial. If $s = 1$, then $G = G_{p_1}$ is a p_1-group, so that

$\mathsf{D}(G) = \mathsf{D}^*(G)$ by the previously mentioned results on the Davenport constant, in which case Proposition 3.3 and our hypotheses, keeping in mind that $n = \exp(G) = \exp(G_{p_1})$, imply

$$\begin{aligned} \mathsf{s}_{n\mathbb{N}}(G) &= \mathsf{D}^*(G \oplus C_n) = \mathsf{D}^*(G_{p_1} \oplus C_n) \\ &= \mathsf{D}^*(G_{p_1}) + n - 1 \le 2\exp(G_{p_1}) - 1 + n - 1 = 3n - 2, \end{aligned}$$

as desired. This completes the base of the induction.

Suppose $s \ge 2$ and the claim holds true for $s-1$. Let $\varphi\colon G \to G/G_{p_s} \cong G_{p_1} \oplus \cdots \oplus G_{p_{s-1}}$ denote the canonical epimorphism. Let $S \in \mathcal{F}(G)$ with $|S| \ge 3n-2$. Let $m = \exp(G_{p_s})$. Since

$$|S| \ge 3n - 2 = (3m-4)n/m + 4n/m - 2$$

and since $\mathsf{s}(G/G_{p_s}) \le 4n/m - 3$ holds by Theorem 4.1.2(b), it follows that S admits a product decomposition $S = S_1 \cdot \ldots \cdot S_{3m-3}S'$ such that each $\varphi(S_i)$ has sum zero and length $|S_i| = n/m$, where $S_1, \ldots, S_{3m-3}, S' \in \mathcal{F}(G)$ (see [18, Lemma 5.7.10]).

In view of $|S'| \ge 3n - 2 - (3m-3)\frac{n}{m} = 3\frac{n}{m} - 2$ and the induction hypothesis, S' has a subsequence S_{3m-2} such that $n/m \mid |S_{3m-2}|$ and $\sigma(\varphi(S_{3m-2})) = 0$. Now, for some generating element $e \in C_n$, let $\iota\colon G \to G \oplus C_n$ denote the map defined via $\iota(g) = g + e$. Then $\sigma(\iota(T)) = \sigma(T) + |T|e$ for each $T \in \mathcal{F}(G)$; in particular, $\sigma(\iota(S_i)) \in G_{p_s} \oplus \langle (n/m)e \rangle$ for each $i \in [1, 3m-2]$. Since

$$\mathsf{D}(G_{p_s} \oplus \langle (n/m)e \rangle) = \mathsf{D}(G_{p_s}) + m - 1 \le 3m - 2,$$

it follows that the sequence $\prod_{i=1}^{3m-2} \sigma(\iota(S_i))$ has a nonempty zero-sum subsequence; let $\emptyset \ne I \subset [1, 3m-2]$ be such that $\sum_{i \in I} \sigma(\iota(S_i)) = 0$. Thus $\sigma(\iota(\prod_{i\in I} S_i)) = \sigma(\prod_{i \in I} S_i) + |\prod_{i\in I} S_i|e = 0$, whence $\prod_{i \in I} S_i$ is a nonempty zero-sum subsequence of S of length divisible by $\operatorname{ord}(e) = n$. □

Parts of the proof of Theorem 4.2 are similar to the proof of Theorem 4.1.

Proof of Theorem 4.2. We state some direct consequences of the assumptions in an explicit form. Let $m = \mathsf{D}(G_q) - \exp(G_q) + 1$. Our assumptions on G imply that there exists some $q \in \mathbb{P}$ and q-group H such that $G \cong H \oplus C_n$ with $\exp(H) \mid n$. Moreover, we know that

$$\mathsf{D}(H) = m$$

divides $\exp(G_q)$, and thus n as well; let $n = mk$. Let $K \cong C_k$ be a subgroup of G such that $G/K \cong H \oplus C_m$. Let $\varphi\colon G \to G/K$ denote the canonical epimorphism. Since m divides $\exp(G_q)$, which is a power of the prime q, it follows that m is itself a power of q. Consequently, since $\exp(H) \le \mathsf{D}^*(H) \le \mathsf{D}(H) = m$ with $\exp(H)$ also a power of the prime q, it follows that

$$\exp(H) \mid m. \tag{4.5}$$

Since H and G_q are both q-groups, so that $\mathsf{D}(H) = \mathsf{D}^*(H)$ and $\mathsf{D}(G_q) = \mathsf{D}^*(G_q)$ (as remarked earlier in the paper), it follows that

$$\mathsf{D}(G_q) - \exp(G_q) = \mathsf{D}(H) - 1. \tag{4.6}$$

We start by establishing the result on $\eta(G)$ and $\mathsf{s}(G)$. On the one hand, by [5, Lemma 3.2] and (4.6), we know

$$\begin{aligned} \eta(G) &\ge 2(\mathsf{D}(H) - 1) + n = 2(\mathsf{D}(G_q) - \exp(G_q)) + n \quad \text{and} \\ \mathsf{s}(G) &\ge 2(\mathsf{D}(H) - 1) + 2n - 1 = 2(\mathsf{D}(G_q) - \exp(G_q)) + 2n - 1. \end{aligned}$$

For the upper bound, first observe that (4.5) implies that $\exp(H \oplus C_m) = m$. In consequence, we have $\exp(H \oplus C_m)\exp(C_k) = mk = n = \exp(G)$. Thus (4.1) and (4.2) imply that

$$\eta(G) \le m(\eta(K) - 1) + \eta(H \oplus C_m) \quad \text{and} \quad \mathsf{s}(G) \le m(\mathsf{s}(K) - 1) + \mathsf{s}(H \oplus C_m). \tag{4.7}$$

Since $K \cong C_k$ is cyclic, we know (see [18, Theorem 5.8.3])

$$\eta(K) = k \quad \text{and} \quad \mathsf{s}(K) = 2k - 1. \tag{4.8}$$

Noting that $H \oplus C_m$ is a q-group with q an odd prime so that (4.5) implies

$$\mathsf{D}(H \oplus C_m) = \mathsf{D}(H) + m - 1 = 2m - 1 = 2\exp(H \oplus C_m) - 1,$$

we see that we can apply Theorem 4.1 to conclude

$$\eta(H \oplus C_m) \le 3m - 2 \text{ and } \mathsf{s}(H \oplus C_m) \le 4m - 3. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) yields

$$\begin{aligned} \eta(G) &\le 2m - 2 + mk = 2(\mathsf{D}(H) - 1) + n \text{ and} \\ \mathsf{s}(G) &\le 2m - 2 + 2mk - 1 = 2(\mathsf{D}(H) - 1) + 2n - 1, \end{aligned}$$

as desired.

It remains to determine $\mathsf{s}_{d\mathbb{N}}(G)$. We continue to use the notation already introduced. By hypothesis, we have $m|d$; as shown above, we also have $G \cong H \oplus C_n$ with $\exp(H)|m$ and $m|n$. Thus it follows, in view of (4.6) and $\mathsf{D}(H) = \mathsf{D}^*(H)$ (as H is a q-group), that

$$\begin{aligned} \mathsf{D}^*(G \oplus C_d) &= \mathsf{D}^*(H) + \gcd(n, d) + \operatorname{lcm}(n, d) - 2 \\ &= \mathsf{D}(G_q) - \exp(G_q) + \gcd(n, d) + \operatorname{lcm}(n, d) - 1. \end{aligned}$$

By Proposition 3.3, we know the above quantity is a lower bound for $\mathsf{s}_{d\mathbb{N}}(G)$. It remains to show it is also an upper bound as well.

Let $S \in \mathcal{F}(G)$ be of the above length $\mathsf{D}^*(G \oplus C_d) = \mathsf{D}^*(H) + \gcd(n, d) + \operatorname{lcm}(n, d) - 2$. As used in the proof for the bounds $\eta(G)$ and $\mathsf{s}(G)$, we know that $\exp(H \oplus C_m) = m$ and

$$\mathsf{s}(H \oplus C_m) \le 4m - 3 \tag{4.10}$$

by Theorem 4.1. We set $j = \gcd(n, d)/m + \operatorname{lcm}(n, d)/m - 2$. Then, recalling that $\mathsf{D}^*(H) = \mathsf{D}(H) = m$, we find that

$$|S| = \mathsf{D}(H) + \gcd(n, d) + \operatorname{lcm}(n, d) - 2 = m(j - 1) + 4m - 2. \tag{4.11}$$

As a result, repeating applying, in view of (4.10), the definition of $\mathsf{s}(H \oplus C_m)$ to $\varphi(S)$ and recalling that $\exp(H \oplus C_m) = m$ in view of (4.5), it follows that S admits a product decomposition $S = S_1 \cdot \ldots \cdot S_j S'$ such that each $\varphi(S_i)$ has sum zero and length $|S_i| = m$, where $S_1, \ldots, S_j, S' \in \mathcal{F}(G)$ (see [18, Lemma 5.7.10]). Since (4.11) implies

$$|S'| = |S| - jm = m(j - 1) + 4m - 2 - jm = 3m - 2$$

and since $\mathsf{s}_{m\mathbb{N}}(H \oplus C_m) \le 3m - 2$ by Theorem 4.1, which we can invoke as explained before (4.9), it follows that S' has a subsequence S_{j+1} with $m \mid |S_{j+1}|$ and $\sigma(S_{j+1}) \in K$.

We consider $\iota\colon G \to G \oplus C_d$ defined via $\iota(g) = g + e$ for some generating element e of C_d. We observe that $\sigma(\iota(S_i)) \in K \oplus \langle me \rangle$ for each $i \in [1, j + 1]$. Since $m \mid d$ and $n = mk$, it follows that

$$K \oplus \langle me_i \rangle \cong C_{n/m} \oplus C_{d/m} \cong C_{\gcd(n,d)/m} \oplus C_{\operatorname{lcm}(n,d)/m}.$$

This is a rank 2 group, so the Davenport constant of this group is $j + 1$ cf. the results mentioned before the proof of Theorem 3.1. Hence the sequence $\prod_{i=1}^{j+1} \sigma(\iota(S_i)) \in \mathcal{F}(K \oplus \langle me_i \rangle)$ has a nonempty zero-sum subsequence. Let $\emptyset \ne I \subset [1, j + 1]$ denote index-set corresponding to this sequence. It follows that $\prod_{i \in I} \iota(S_i) \in \mathcal{F}(G \oplus C_d)$ is a zero-sum sequence, whence $\prod_{i \in I} S_i \in \mathcal{F}(G)$ is a zero-sum subsequence of S with length divisible by d (by the same arguments used at the end of the proof of Theorem 4.1). □

We end this section by discussing the relevance of the assumptions in our results.

Remark 4.3.

1. It is conceivable that the assumption $\mathsf{D}(G_q) - \exp(G_q) + 1 \mid \exp(G_q)$ in Theorem 4.2 can actually be replaced by the assumption $\mathsf{D}(G_q) - \exp(G_q) + 1 \le \exp(G_q)$ of Theorem 4.1. We could relax the assumption in this way if [30, Conjecture 4.1] were true; this conjecture concerns the exact value of $\eta(G_q)$ and $\mathsf{s}(G_q)$ under this weaker assumption.

2. The restriction that $\exp(G)$ and q are odd, which is imposed in the second part of our result, is due to the fact that [30, Theorem 1.2] is only applicable in this case, yet it is well possible that the statement holds for 2-groups as well, in which case these assumptions could be dropped (cf. again [30, Conjecture 4.1]).
3. The restriction in Theorem 4.2 that G/G_q is cyclic is very likely not technical. It seems quite unlikely that there is a uniform argument of this form to determine the precise value of the constants under the assumptions of Theorem 4.1. For example, note that in this more general setting, $\mathsf{D}^*(G)$ depends on the *precise* structure of each of the p-subgroups of G (also see the results in [5]). Yet, imposing the assumption that G is a group of rank 2, and thus each p-subgroup has at most rank 2, the values of $\mathsf{D}(G)$, $\eta(G)$, and $\mathsf{s}(G)$ are known, and we additionally determine $\mathsf{s}_{d\mathbb{N}}(G)$ for any d (see Section 5).

5. On $\mathsf{s}_{d\mathbb{N}}(G)$ for groups of rank two

In this section, we determine $\mathsf{s}_{d\mathbb{N}}(G)$ for rank 2 groups G. To this end, we again use the inductive method. However, in contrast to essentially all earlier applications of the inductive method to direct zero-sum problems, there is an additional complication. Typically, once, for an appropriately chosen subgroup, the 'relative' problems in the subgroup and the factor group are solved, the final step of recombining these subsequences from the 'relative' problems to yield the desired subsequence in the original group is straightforward (cf. the proof of Theorem 4.1 for an example). Yet, for this problem, this is not so, and an additional argument is needed.

For the proof, we make use of the facts that

$$\mathsf{s}(C_m \oplus C_n) = 2n + 2m - 3 \tag{5.1}$$

and that

$$\mathsf{D}(C_m \oplus C_n) = m + n - 1 \tag{5.2}$$

when $1 \le m \mid n$ (see [18, Theorem 5.8.3]). In addition we need the following lemma, which makes use of the fact that the gap between $\mathsf{s}_{n\mathbb{N}}(G)$ and $\mathsf{s}(G)$, for G a group of rank two, is not too large.

Lemma 5.1. *Let $G \cong C_m \oplus C_n$ with $1 \le m \mid n$, and let $t \in \mathbb{N}$. If $S \in \mathcal{F}(G)$ is a sequence with*

$$|S| \ge (t-1)n + \mathsf{s}_{n\mathbb{N}}(G),$$

then S has a decomposition $S = S_1 \cdot \ldots \cdot S_t S'$ with each S_i zero-sum, $|S_i| = n$ for $i \in [1, t-1]$, and $|S_t| \in \{n, 2n\}$, where $S_1, \ldots, S_n, S' \in \mathcal{F}(G)$.

Proof. In view of Theorem 4.1.1(b), we know that $|S''| \ge \mathsf{s}_{n\mathbb{N}}(G)$ implies that $S'' \in \mathcal{F}(G)$ contains a zero-sum sequence of length n or $2n$. From Proposition 3.3 and Lemma 3.2, we know

$$\mathsf{s}_{n\mathbb{N}}(G) \ge \mathsf{D}^*(G \oplus C_n) = n + \mathsf{D}^*(G) - 1 = 2n + m - 2. \tag{5.3}$$

From (5.1), we know

$$\mathsf{s}(G) \le 2n + 2m - 3. \tag{5.4}$$

In view of (5.3) and (5.4), we have $n + \mathsf{s}_{n\mathbb{N}}(G) \ge 3n + m - 2 \ge \mathsf{s}(G)$. Thus, in view of $|S| \ge (t-1)n + \mathsf{s}_{n\mathbb{N}}(G)$, we can repeatedly apply the definition of $\mathsf{s}(G)$ to S to find $t-1$ zero-sum subsequences $S_1, \ldots, S_{t-1}$ with $S_1 \cdot \ldots \cdot S_{t-1} \mid S$ and $|S_i| = n$ for all i. Let $S'' = S(S_1 \cdot \ldots \cdot S_{t-1})^{-1}$. Then $|S''| = |S| - (t-1)n \ge \mathsf{s}_{n\mathbb{N}}(G)$. Hence, as remarked at the beginning of the proof, S'' must have a zero-sum subsequence S_t with $|S_t| = n$ or $|S_t| = 2n$, completing the proof. □

Theorem 5.2. *Let $d \in \mathbb{N}$ and let $G \cong C_m \oplus C_n$ with $1 \le m \mid n$. Then*

$$\mathsf{s}_{d\mathbb{N}}(G) = \mathsf{D}^*(G \oplus C_d) = \operatorname{lcm}(n,d) + \gcd(n, \operatorname{lcm}(m,d)) + \gcd(m,d) - 2.$$

Proof. When $m = 1$, this follows from Theorem 3.1.1. Therefore we assume $m > 1$. Since G has rank two, it follows that each p-component G_p has rank at most two, and thus $\mathsf{D}(G_p) = \mathsf{D}^*(G_p) \le 2\exp(G_p) - 1$ for all primes dividing n. Note that Lemma 3.2 implies that

$$\mathsf{D}^*(G \oplus C_d) = \operatorname{lcm}(n,d) + \gcd(n, \operatorname{lcm}(m,d)) + \gcd(m,d) - 2, \tag{5.5}$$

while Proposition 3.3 shows that this is a lower bound for $\mathsf{s}_{d\mathbb{N}}(G)$. It remains to show it is also an upper bound. We begin by considering two particular cases, whose proof is similar to existing arguments. The main novelty comes in the final arguments that assemble the information found in these special cases, which are also auxiliary results.

Case 1: $d = n$. If $m = n$, then (5.5) becomes $\mathsf{D}^*(G \oplus C_d) = 3n - 2$, and the result follows from Theorem 4.1.1(b). Therefore we assume $m < n$. We proceed by a minor modification of the argument used for Theorem 4.1.1(b). Since $m < n$, let $n = km$ with $k \ge 2$. Let $K \le G$ be a subgroup such that

$$K \cong C_k \quad \text{and} \quad G/K \cong C_m \oplus C_m$$

and let $\varphi : G \to G/K \cong C_m \oplus C_m$ denote the natural homomorphism. Note, under the assumption $d = n$, that (5.5) becomes

$$\mathsf{D}^*(G \oplus C_d) = 2n + m - 2.$$

Let $S \in \mathcal{F}(G)$ with $|S| = 2n + m - 2$. By the previously handled case ($d = m = n$), it follows that $\mathsf{s}_{m\mathbb{N}}(G/K) = \mathsf{D}^*(G/K \oplus C_m) = 3m - 2$. Thus

$$|\varphi(S)| = |S| = (2k-2)m + 3m - 2 = (2k-2)m + \mathsf{s}_{m\mathbb{N}}(G/K).$$

Applying Lemma 5.1 to $\varphi(S)$, we find a product decomposition $S = S_1 \cdot \ldots \cdot S_{2k-1} S'$ with each S_i being zero-sum modulo K and of length $|S_i| \in \{m, 2m\}$. Let $\iota : G \to G \oplus \langle e \rangle \cong G \oplus C_n$, where $\operatorname{ord}(e) = n$, be the map defined by letting $\iota(g) = g + e$. Then, since each S_i is zero-sum modulo K with length a multiple of m, it follows that $\sigma(\iota(S_i)) \in K \oplus \langle me \rangle \cong C_k \oplus C_k$ for each $i \in [1, 2k-1]$. Since $\mathsf{D}(C_k \oplus C_k) = 2k - 1$ by (5.2), applying the definition of $\mathsf{D}(C_k \oplus C_k)$ to the sequence $\prod_{i=1}^{2k-1} \sigma(\iota(S_i)) \in \mathcal{F}(K \oplus \langle me \rangle)$ yields a nonempty zero-sum sequence, say indexed by $I \subset [1, 2k-1]$. Thus $0 = \sigma(\prod_{i \in I} \iota(S_i)) = \sigma(\prod_{i \in I} S_i) + |\prod_{i \in I} S_i| e$, whence $\prod_{i \in I} S_i \in \mathcal{F}(G)$ is a nonempty zero-sum subsequence of S whose length is divisible by $\operatorname{ord}(e) = n$, as desired. This completes the case $d = n$.

Case 2: $d \mid n$. Let $u = \frac{\operatorname{lcm}(m,d)}{d} = \frac{m}{\gcd(m,d)}$ and $v = \frac{n}{\operatorname{lcm}(m,d)}$. Note $uvd = n$. Let $K \le G$ be a subgroup such that

$$K \cong C_u \oplus C_{uv} \quad \text{and} \quad G/K \cong C_{\gcd(m,d)} \oplus C_d$$

and let $\varphi : G \to G/K \cong C_{\gcd(m,d)} \oplus C_d$ denote the natural homomorphism. Note, under the assumption $d \mid n$, that (5.5) becomes

$$\mathsf{D}^*(G \oplus C_d) = n + \operatorname{lcm}(m,d) + \gcd(m,d) - 2. \tag{5.6}$$

Let $S \in \mathcal{F}(G)$ be a sequence with

$$|S| = n + \operatorname{lcm}(m,d) + \gcd(m,d) - 2 = (uv + u - 2)d + 2d + \gcd(m,d) - 2.$$

In view of Case 1 and (5.5), we have $\mathsf{s}_{d\mathbb{N}}(G/K) = \mathsf{D}^*(G/K \oplus C_d) = 2d + \gcd(m,d) - 2$. Thus, applying Lemma 5.1 to $\varphi(S)$, we find a product decomposition $S = S_1 \cdot \ldots \cdot S_{uv+u-1} S'$ with each S_i zero-sum modulo K and of length divisible by d. But now, in view of (5.2), the sequence $\prod_{i=1}^{uv+u-1} \sigma(S_i) \in \mathcal{F}(K)$ has length $uv + u - 1 = \mathsf{D}(C_u \oplus C_{uv}) = \mathsf{D}(K)$. Hence, applying the definition of $\mathsf{D}(K)$ to $\prod_{i=1}^{uv+u-1} \sigma(S_i)$, we find a non-empty zero-sum subsequence, say indexed by $I \subset [1, uv + u - 1]$. Thus $\sigma(\prod_{i \in I} S_i) = 0$. Moreover, since $d \mid |S_i|$ for each i, it follows that $d \mid |\prod_{i \in I} S_i|$, as desired. This completes the case $d \mid n$.

We now proceed to show

$$\mathsf{s}_{d\mathbb{N}}(G) \le \operatorname{lcm}(n,d) - n + \mathsf{s}_{\gcd(n,d)\mathbb{N}}(G). \tag{5.7}$$

Once (5.7) is established, then, applying Case 2 to $\mathsf{s}_{\gcd(n,d)\mathbb{N}}(G)$ and using (5.6), we will know

$$\begin{aligned} \mathsf{s}_{d\mathbb{N}}(G) &\le \operatorname{lcm}(n,d) - n + \mathsf{D}^*(G \oplus C_{\gcd(n,d)}) \\ &= \operatorname{lcm}(n,d) - n + (n + \operatorname{lcm}(m, \gcd(n,d)) + \gcd(m, \gcd(n,d)) - 2) \\ &= \operatorname{lcm}(n,d) + \operatorname{lcm}(m, \gcd(n,d)) + \gcd(m,d) - 2, \end{aligned}$$

which is equal to $\mathsf{D}^*(G \oplus C_d)$ by Lemma 3.2. In consequence, once (5.7) is established, the proof will be complete. We continue with the proof of (5.7). As (5.7) holds trivially when $d \mid n$, we assume $d \nmid n$.

Let $\alpha n = \operatorname{lcm}(n,d)$. Then, since $d \nmid n$, we have $\alpha \ge 2$. Let $S \in \mathcal{F}(G)$ be a sequence with

$$|S| = \operatorname{lcm}(n,d) - n + \mathsf{s}_{\gcd(n,d)\mathbb{N}}(G) = (\alpha - 1)n + \mathsf{s}_{\gcd(n,d)\mathbb{N}}(G). \tag{5.8}$$

By Case 2 and (5.5), we have

$$\mathsf{s}_{\gcd(n,d)\mathbb{N}}(G) = n + \operatorname{lcm}(m, \gcd(n,d)) + \gcd(m, \gcd(n,d)) - 2 \ge n + m - 1. \tag{5.9}$$

Thus it follows from (5.1) that

$$2n + \mathsf{s}_{\gcd(n,d)\mathbb{N}}(G) \ge 3n + m - 1 \ge \mathsf{s}(G).$$

Consequently, in view of (5.8) and $\alpha \ge 2$, it follows, by repeatedly applying the definition of $\mathsf{s}(G)$ to S, that we can find $\alpha - 2$ zero-sum subsequences $S_1, \ldots, S_{\alpha-2} \in \mathcal{F}(G)$ such that $S_1 \cdot \ldots \cdot S_{\alpha-2} | S$ and $|S_i| = n$ for all $i \in [1, \alpha - 2]$. Let $S' = S(S_1 \cdot \ldots \cdot S_{\alpha-2})^{-1}$. Then, in view of (5.9) and Case 1, we have

$$|S'| = |S| - (\alpha - 2)n = n + \mathsf{s}_{\gcd(n,d)\mathbb{N}}(G) \ge 2n + m - 1 \ge \mathsf{s}_{n\mathbb{N}}(G).$$

Hence, since $\mathsf{s}_{n\mathbb{N}}(G) < 3n$, applying the definition of $\mathsf{s}_{n\mathbb{N}}(G)$ to S' yields a zero-sum subsequence $S_{\alpha-1} \mid S'$ with $|S_{\alpha-1}| = n$ or $|S_{\alpha-1}| = 2n$. If $|S_{\alpha-1}| = 2n$, then $S_1 \cdot \ldots \cdot S_{\alpha-1}$ is a zero-sum subsequence of S with length $(\alpha - 2)n + 2n = \alpha n = \operatorname{lcm}(n,d)$, which is a multiple of d, and thus of the desired length. Therefore we may instead assume $|S_{\alpha-1}| = n$. Let $S'' = S(S_1 \cdot \ldots \cdot S_{\alpha-1})^{-1}$. Then $|S''| = |S| - (\alpha-1)n = \mathsf{s}_{\gcd(n,d)\mathbb{N}}(G)$, so applying the definition of $\mathsf{s}_{\gcd(n,d)\mathbb{N}}(G)$ to S'' yields a zero-sum sequence $S_0 \mid S''$ with length $|S_0| = k_0 \gcd(n,d)$ for some $k_0 \in \mathbb{N}$.

Since $\alpha n = \operatorname{lcm}(n,d)$, it follows that

$$d = \alpha \gcd(n,d).$$

Let $n = n' \gcd(n,d)$. Then, since $d = \alpha \gcd(n,d)$, we see that

$$\gcd(\alpha, n') = 1.$$

If $k_0 \equiv 0 \mod \alpha$, then

$$|S_0| = k_0 \gcd(n,d) \equiv \alpha \gcd(n,d) \equiv 0 \mod \alpha \gcd(n,d),$$

in which case, since $\alpha \gcd(n,d) = d$, we see that S_0 is a zero-sum subsequence of length divisible by d, as desired. Therefore we may assume $k_0 \not\equiv 0 \mod \alpha$.

Observe that

$$|S_1 \cdot \ldots \cdot S_j| = jn = jn' \gcd(n,d) \quad \text{for } j \in [1, \alpha - 1].$$

Thus, since $\gcd(\alpha, n') = 1$, we conclude that

$$\{\frac{1}{\gcd(n,d)}|S_1|, \frac{1}{\gcd(n,d)}|S_1 S_2|, \ldots, \frac{1}{\gcd(n,d)}|S_1 \cdot \ldots \cdot S_{\alpha-1}|\}$$

is a full set of nonzero residue classes modulo α. Consequently, since $k_0 \not\equiv 0 \mod \alpha$, we can find $k \in [1, \alpha - 1]$ such that $\frac{1}{\gcd(n,d)}|S_1 \cdot \ldots \cdot S_k| + k_0 \equiv 0 \mod \alpha$, in which case

$$|S_0 S_1 \cdot \ldots \cdot S_k| = |S_1 \cdot \ldots \cdot S_k| + k_0 \gcd(n,d) \equiv 0 \mod \alpha \gcd(n,d).$$

Since $\alpha \gcd(n,d) = d$, this means that $S_0 S_1 \cdot \ldots \cdot S_k$ is a subsequence of S with length divisible by d. Moreover, since each S_i is zero-sum, it follows that the subsequence $S_0 S_1 \cdot \ldots \cdot S_k$ is also zero-sum, whence we have found a zero-sum sequence of the desired length, completing the proof of (5.7), which completes the proof as remarked earlier. □

6. Upper bounds for the lengths of zero-sum subsequences

Let H be a Krull monoid with class group G and suppose that every class contains a prime divisor. The investigation of sets of lengths of the form $\mathsf{L}(uv)$, where $u, v \in H$ are irreducible elements, is a frequently studied topic in the theory of non-unique factorizations (see for example [18, Section 6.6]). Only recently, a close connection of this topic with the catenary degree $\mathsf{c}(H)$ of H was found—see the invariant $\daleth(H)$ introduced in [17]. As is well-known, the study of sets $\mathsf{L}(uv)$ translates into a zero-sum problem as follows: pick two minimal zero-sum sequences U and V over G and find product decompositions of the form $UV = W_1 \cdot \ldots \cdot W_k$ with $W_1, \ldots, W_k$ minimal zero-sum sequences over G. To control the number k of atoms in such a factorization, it is desirable to be able to find zero-sum subsequences of the (long) zero-sum sequence UV with bounded lengths (see Condition (b) in Lemma 6.1).

Thus, in zero-sum terminology, we have to study conditions which imply that, for a given $d \in [1, \mathsf{D}(G)-1]$, every zero-sum sequence $A \in \mathcal{F}(G)$ of length $|A| \ge \mathsf{D}(G)+1$ has a zero-sum subsequence T of length $|T| \in [1, d]$. Since, by definition, $\mathsf{D}(G)$ is the maximal length of a minimal zero-sum sequence, it makes no sense to consider the above question for sequences A of length less than $\mathsf{D}(G)+1$. We start with a simple characterization of this property which allows us to obtain a natural restriction for d; the optimality of this condition is discussed in the remark below.

Lemma 6.1. *Let $d \in \mathbb{N}$ with $\mathsf{D}(G) \le 2d-1$. Then the following statements are equivalent:*

(a) *For all $U, V \in \mathcal{A}(G)$ with $|UV| \ge 2d$, there exists a zero-sum subsequence T of UV of length $|T| \in [1, d]$.*
(b) *For all $U, V \in \mathcal{A}(G)$, there exists a zero-sum subsequence T of UV of length $|T| \in [1, d]$.*
(c) *Every zero-sum sequence $A \in \mathcal{F}(G)$ of length $|A| \ge \mathsf{D}(G)+1$ has a zero-sum subsequence T of length $|T| \in [1, d]$.*

Proof. (a) ⇒ (b) Let $U, V \in \mathcal{A}(G)$ be given, say $|U| \le |V|$. If $|UV| \ge 2d$, then the assertion follows from (a). If $|UV| \le 2d$, then we set $T = U$ and get $2|T| \le |UV| \le 2d$.

(b) ⇒ (c) Let $A \in \mathcal{F}(G)$ be zero-sum. Then there are $U_1, \ldots, U_k \in \mathcal{A}(G)$ such that $A = U_1 \cdot \ldots \cdot U_k$. Since $|A| \ge \mathsf{D}(G)+1$, it follows that $k \ge 2$. Thus there exists a zero-sum sequence T with $T \mid U_1U_2$, and hence with $T \mid A$ also, such that $|T| \in [1, d]$.

(c) ⇒ (a) Obvious. □

Remark 6.2. Let $d \in \mathbb{N}$. In general, none of the statements in the previous lemma can hold without the assumption $\mathsf{D}(G) \le 2d-1$. This can be seen from the following example. Take $G = H \oplus H$ such that $\mathsf{D}(G) = 2\mathsf{D}(H) - 1$ (note that this holds true if H is cyclic or a p-group). Then there are $U, V \in \mathcal{A}(G)$ such that $\langle \mathrm{supp}(U) \rangle \cap \langle \mathrm{supp}(V) \rangle = \{0\}$ and $|U| = |V| = \mathsf{D}(H)$. Thus the only nonempty zero-sum subsequences of UV are U and V, which have length

$$|U| = |V| = \frac{\mathsf{D}(G)+1}{2}\,.$$

We give the main result of this section; see below for groups fulfilling the assumptions.

Theorem 6.3. *Let $d \in \mathbb{N}$ with $\mathsf{D}(G) \le 2d-1$ and suppose that $\mathsf{s}_{d\mathbb{N}}(G) \le 3d-1$.*

1. *Every sequence $S \in \mathcal{F}(G)$ of length $|S| = \mathsf{s}_{d\mathbb{N}}(G)$ has a zero-sum subsequence T of length $|T| \in [1, d]$.*

2. *Every zero-sum sequence* $A \in \mathcal{F}(G)$ *of length* $|A| \ge \mathsf{D}(G) + 1$ *has a zero-sum subsequence* T *of length* $|T| \in [1, d]$.

Proof. 1. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = \mathsf{s}_{d\mathbb{N}}(G)$. Since $\mathsf{s}_{d\mathbb{N}}(G) \le 3d - 1$, S has a zero-sum subsequence T of length $|T| \in \{d, 2d\}$. If $|T| = d$, then we are done. If $|T| = 2d$, then $2d \ge \mathsf{D}(G) + 1$ implies that T has a product decomposition $T = T_1 T_2$ with T_1 and T_2 nonempty zero-sum sequences. Clearly, we have $\min\{|T_1|, |T_2|\} \in [1, d]$.

2. Let $A \in \mathcal{F}(G)$ be zero-sum with $|A| \ge \mathsf{D}(G) + 1$. Then A is a product of two nonempty zero-sum subsequences, and if $|A| \le 2d$, then the assertion is clear as before. Suppose that $|A| \ge 2d + 1$. If $|A| \ge \mathsf{s}_{d\mathbb{N}}(G)$, then the assertion follows from 1. Therefore we have

$$2d + 1 \le |A| < \mathsf{s}_{d\mathbb{N}}(G) \le 3d - 1. \tag{6.1}$$

Now the sequence

$$S = 0^k A, \quad \text{where} \quad k = \mathsf{s}_{d\mathbb{N}}(G) - |A| \in [1, d - 2],$$

has a zero-sum subsequence $T = 0^{k'} A'$ of length $|T| \in \{d, 2d\}$, where $k' \in [0, k]$ and $A' \mid A$. If $|T| = d$, then A' is a zero-sum subsequence of A of length $|A'| \in [1, d]$, as desired. If $|T| = 2d$, then A' is a zero-sum subsequence of length

$$|A'| \ge 2d - k = 2d + |A| - \mathsf{s}_{d\mathbb{N}}(G).$$

Hence, ${A'}^{-1}A$ is a zero-sum subsequence (as both A and A' are zero-sum sequences) with length (in view of (6.1))

$$\begin{aligned} |{A'}^{-1}A| &= |A| - |A'| \le |A| - (2d + |A| - \mathsf{s}_{d\mathbb{N}}(G)) \\ &= \mathsf{s}_{d\mathbb{N}}(G) - 2d \le 3d - 1 - 2d = d - 1. \end{aligned}$$

Moreover, since (6.1) implies $|A| \ge 2d + 1$ while $A' \mid T$ implies $|A'| \le |T| = 2d$, we see that ${A'}^{-1}A$ is also a nonempty zero-sum subsequence, and the proof is complete in this case as well. □

Results of the two preceding sections yield various classes of groups fulfilling the conditions of Theorem 6.3. The groups covered by the assumptions of Theorem 4.1.1, thus in particular groups of rank two, fulfil the conditions of Corollary 6.4. In the special case of groups of rank two, the result was first achieved in [16, Lemma 3.6].

Corollary 6.4. *Let* $\exp(G) = n$ *and suppose that* $\mathsf{D}(G) \le 2n - 1$ *and* $\mathsf{s}_{n\mathbb{N}}(G) \le 3n - 1$. *Then every zero-sum sequence* $A \in \mathcal{F}(G)$ *of length* $|A| \ge \mathsf{D}(G) + 1$ *has a nonempty zero-sum subsequence of length at most* $\exp(G)$.

Proof. This is a special case of Theorem 6.3.2. □

Corollary 6.5. *Let* G *be a* p*-group. Suppose there exists some* $i \in [1, \mathsf{D}(G)]$ *such that* $(\mathsf{D}^*(G) + i)/2$ *is a power of* p. *Then every zero-sum sequence* $A \in \mathcal{F}(G)$ *of length* $|A| \ge \mathsf{D}(G) + 1$ *has a zero-sum subsequence* T *of length* $|T| \in [1, (\mathsf{D}^*(G) + i)/2]$.

Proof. We set $d = (\mathsf{D}^*(G) + i)/2$. Then $2d = \mathsf{D}^*(G) + i \ge \mathsf{D}(G) + 1$, and thus Theorem 3.1.2(a) implies that $\mathsf{s}_{d\mathbb{N}}(G) \le \mathsf{D}^*(G) + d - 1 \le \mathsf{D}(G) + d - 1 \le 3d - 2$. Therefore the assertion follows from Theorem 6.3. □

Note, if $(\mathsf{D}^*(G) + 1)/2$ is a power of p, then the above result is best possible, as can be seen from the example discussed in Remark 6.2.

References

[1] S.D. Adhikari, D.J. Grynkiewicz, and Zhi-Wei Sun, *On weighted zero-sum sequences*, manuscript.

[2] G. Bhowmik and J.-C. Schlage-Puchta, *Davenport's constant for groups of the form* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, Additive Combinatorics (A. Granville, M.B. Nathanson, and J. Solymosi, eds.), CRM Proceedings and Lecture Notes, vol. 43, American Mathematical Society, 2007, pp. 307 – 326.

[3] R. Chi, S. Ding, W. Gao, A. Geroldinger, and W.A. Schmid, *On zero-sum subsequences of restricted size IV*, Acta Math. Hung. **107** (2005), 337 – 344.

[4] C. Delorme, O. Ordaz, and D. Quiroz, *Some remarks on Davenport constant*, Discrete Math. **237** (2001), 119 – 128.

[5] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, and L. Rackham, *Zero-sum problems in finite abelian groups and affine caps*, Quarterly. J. Math., Oxford II. Ser. **58** (2007), 159 – 186.

[6] M. Freeze and W.A. Schmid, *Remarks on a generalization of the Davenport constant*, Discrete Math. **310** (2010), 3373 – 3389.

[7] W. Gao, *On zero sum subsequences of restricted size III*, Ars Comb. **61** (2001), 65 – 72.

[8] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. **24** (2006), 337 – 369.

[9] ———, *On the number of subsequences with given sum of sequences over finite abelian p-groups*, Rocky Mt. J. Math. **37** (2007), 1541 – 1550.

[10] W. Gao, A. Geroldinger, and W.A. Schmid, *Inverse zero-sum problems*, Acta Arith. **128** (2007), 245 – 279.

[11] W. Gao, Y. ould Hamidoune, and G. Wang, *Distinct lengths modular zero-sum subsequences: a proof of Graham's conjecture*, J. Number Theory **130** (2010), 1425 – 1431.

[12] W. Gao and J. Peng, *On the number of zero-sum subsequences of restricted size*, Integers **9** (2009), Paper A41, 537 – 554.

[13] W. Gao and R. Thangadurai, *On zero-sum sequences of prescribed length*, Aequationes Math. **72** (2006), 201 – 212.

[14] A. Geroldinger, *On a conjecture of Kleitman and Lemke*, J. Number Theory **44** (1993), 60 – 65.

[15] ———, *Additive group theory and non-unique factorizations*, Combinatorial Number Theory and Additive Group Theory (A. Geroldinger and I. Ruzsa, eds.), Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009, pp. 1 – 86.

[16] A. Geroldinger and D.J. Grynkiewicz, *On the structure of minimal zero-sum sequences with maximal cross number*, J. Combinatorics and Number Theory **1 (2)** (2009), 9 – 26.

[17] A. Geroldinger, D.J. Grynkiewicz, and W.A. Schmid, *The catenary degree of Krull monoids I*, manuscript.

[18] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.

[19] A. Geroldinger, M. Liebmann, and A. Philipp, *On the Davenport constant and on the structure of extremal sequences*, Period. Math. Hung., to appear.

[20] A. Geroldinger and R. Schneider, *On Davenport's constant*, J. Comb. Theory, Ser. A **61** (1992), 147 – 152.

[21] B. Girard, *On the existence of distinct lengths zero-sum subsequences*, Rocky Mt. J. Math., to appear.

[22] D.J. Grynkiewicz, *On a conjecture of Hamidoune for subsequence sums*, Integers **5(2)** (2005), Paper A07, 11p.

[23] ———, *On extending Pollard's theorem for t-representable sums*, Isr. J. Math. **177** (2010), 413 – 440.

[24] S.S. Kannan and S.K. Pattan, *Projective normality of finite group quotients and EGZ theorem.*

[25] S.S. Kannan, S.K. Pattanayak, and P. Sardar, *Projective normality of finite group quotients*, Proc. Am. Math. Soc. **137** (2009), 863 – 867.

[26] S. Kubertin, *Nullsummen in* $\mathbb{Z}_p^d$, Master's thesis, Technical University Clausthal, 2002.

[27] C. Reiher, *On Kemnitz' conjecture concerning lattice points in the plane*, Ramanujan J. **13** (2007), 333 – 337.

[28] W.A. Schmid, *The inverse problem associated to the Davenport constant for* $C_2 \oplus C_2 \oplus C_{2n}$*, and applications to the arithmetical characterization of class groups*, submitted.

[29] ———, *On zero-sum subsequences in finite abelian groups*, Integers **1** (2001), Paper A01, 8p.

[30] W.A. Schmid and J.J. Zhuang, *On short zero-sum subsequences over p-groups*, Ars Comb. **95** (2010), 343 – 352.

[31] P. Yuan, H. Guan, and X. Zeng, *Normal sequences over finite abelian groups*, manuscript.

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