Abstract. Let $A, B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets each covered by precisely 2 horizontal lines. Suppose $\langle A + B - A - B \rangle = \mathbb{Z}^2$, $|A| \geq |B|$ and $|A + B| = |A| + 2|B| - 3 + r \leq |A| + \frac{|B|}{2} - 5$. Then there exist subsets $P_A, P_B, P \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference $(1, 0)$, such that $A \subseteq P_A$, $B \subseteq P_B$ and $(x + A) \cup (y + B) \subseteq P$, for some $x, y \in \mathbb{Z}^2$, with $|P_A| \leq |A| + r$, $|P_B| \leq |B| + r$, $|P_A| + |P_B| \leq 2|B| + 2r$ and $|P| \leq \frac{|A| + |B|}{2} + \frac{3}{2}r$. A similar result is proved assuming $A$ is covered by 2 horizontal lines and $B$ by 1 and vice versa. This generalizes a result of Stanchescu handling the case $A = B$ and extends the Freiman $3k - 4$ Theorem to 2-dimensional sumsets with $|A + B| < |A| + \frac{7}{2}|B| - 5$.

1. Introduction

Let $G$ be an abelian group and let $A, B \subseteq G$ be subsets of $G$. We define their sumset to be

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

Moreover, we let

$$\delta(A, B) = \begin{cases} 
1 & \text{if } x + A \subseteq B \text{ for some } x \in G, \\
0 & \text{otherwise.}
\end{cases}$$

We let $\langle A \rangle$ denote the subgroup generated by $A$ and observe that $\langle A - A \rangle$ is the minimal subgroup $K$ such that $A$ is contained in a $K$-coset. In particular, if $0 \in A$, then $\langle A - A \rangle = \langle A \rangle$. Likewise, for $G = \mathbb{Z}$, $\gcd(A - A) \in \mathbb{N}_0$ is the minimal non-negative integer $d$ such that $A$ is contained in an arithmetic progression with difference $d$, with $\gcd(A - A) = \gcd(A)$ when $0 \in A$. We let

$$\text{diam}(A) = \max A - \min A$$

denote the diameter of a finite subset $A \subseteq \mathbb{Z}$. If $G = \mathbb{Z}^d$, we let $e_1, \ldots, e_d \in \mathbb{Z}^d$ be the standard basis vectors, so $e_i$ has a 1 at the $i$-th coordinate and zeros elsewhere, and we let $\pi_i : \mathbb{Z}^d \to \mathbb{Z}$ be the projection onto the $i$-th coordinate with respect to the basis $e_1, \ldots, e_d \in \mathbb{Z}^d$.

Our starting point is the $3k - 4$ Theorem. The following is the most general version of the $3k - 4$ Theorem currently known. The form given below may be found in [6, Theorem 7.1 and comments thereafter] and is the result of successive contributions from Freiman [3], Lev and Sneliansky [8], Stanchescu [12], and Bardaji and Grynkiewicz [1]. Worth noting, if $|B| \leq |A|$, then the upper bound from Theorem A(i) becomes

$$|A + B| \leq |A| + 2|B| - 3 - \delta(A, B).$$
Indeed, \(|A| - \delta(B, A) < |B| - \delta(A, B)| is only possible, in view of \(|A| \geq |B|\), if \(|A| = |B|\), \(\delta(A, B) = 0\) and \(\delta(B, A) = 1\). However, it is easily noted from the definition of \(\delta\) that \(\delta(A, B) = \delta(B, A)\) when \(|A| = |B|\), meaning this is actually never possible. Also, both hypotheses (i) and (ii) imply \(|A|, |B| \geq 2\) (equivalent to \(\text{diam } A, \text{diam } B \geq 1\)) in view of basic lower bounds for \(|A + B|\) (see Theorem C).

**Theorem A** (3k−4 Theorem). Let \(A, B \subseteq \mathbb{Z}\) be finite and nonempty, let \(d = \gcd(A + B - A - B)\) and let \(|A + B| = |A| + |B| - 1 + r\). If either

(i) \(|A + B| \leq |A| + |B| - 3 + \min\{ |B| - \delta(A, B), |A| - \delta(B, A) \}\), or

(ii) \(\text{diam } B \leq \text{diam } A\), \(\gcd(A - A) \leq 2d\) and \(|A + B| \leq |A| + 2|B| - 3 - \delta(A, B)\),

then there are arithmetic progressions \(P, Q\) and \(R\) of common difference \(d\) with

\[
A \subseteq P, \quad B \subseteq Q, \quad |P \setminus A| \leq r, \quad |Q \setminus B| \leq r, \quad (1)
\]

\[R \subseteq A + B \text{ and } |R| \geq |A| + |B| - 1.\]

Moreover, if either (i) or (ii) holds with \(\text{diam } B \leq \text{diam } A\) and \(|B| \geq |A|\), then

\[|Q \setminus B| \leq r - (|B| - |A|).\]

In short, the 3k−4 theorem shows that a sumset \(A + B \subseteq \mathbb{Z}\) with small sumset below the threshold \(|A + B| \leq |A| + |B| - 3 + \min\{ |B| - \delta(A, B), |A| - \delta(B, A) \}\) is only possible if \(A, B\) and \(\mathbb{Z} \setminus (A + B)\) can all be approximated by arithmetic progressions \(P, Q\) and \(R\) with common difference, in the sense that the “distance” between each set and the respective set \(P, Q\) and \(\mathbb{Z} \setminus R\), as measured by inclusion, is small. Note that the set \(A + B\) containing a long arithmetic progression \(R\) is equivalent to its complement being contained in the complement of \(R\) with \(|(\mathbb{Z} \setminus (A + B)) \setminus (\mathbb{Z} \setminus R)| \leq r\). Thus the theorem is symmetric with regards to all three sets \(A, B\) and \(A + B\). Moreover, the bound \(r\) for the number of holes separating the individual sets from their progressions is known to be precise, as is the threshold hypothesis \(|A + B| \leq |A| + |B| - 3 + \min\{ |B| - \delta(A, B), |A| - \delta(B, A) \}\). In both cases, there are examples showing these bounds cannot be improved. For instance, if \(r \geq 0, a \geq r + 1\) and \(b \geq r + 1\) with strict inequality in at least one of the latter two inequalities, then the sets

\[
A = [0, a - 1 - r] \cup \{a - r + 1, a - r + 3, \ldots, a - r + (2r - 1)\} \quad \text{and} \quad B = [0, b - 1 - r] \cup \{b - r + 1, b - r + 3, \ldots, b - r + (2r - 1)\}
\]

have \(|A| = a, |B| = b\), and

\[
A + B = [0, a + b - 2] \cup \{a + b, a + b + 2, \ldots, a + b + 2r - 2\}
\]

with \(|A + B| = |A| + |B| - 1 + r \leq |A| + |B| - 2 + \min\{ |B| - \delta(A, B), |A| - \delta(B, A) \}\), and it is easily seen that \(|P \setminus A| = |Q \setminus B| = |(\mathbb{Z} \setminus R) \setminus (\mathbb{Z} \setminus (A + B))| = r\), where \(P = [0, a + 1 - r]\), \(Q = [0, b - 1 - r]\) and \(R = [0, a + b - 2]\), cannot be improved upon. Likewise, if \(A = I_1 \cup I_2\) is the union of two intervals \(I_1\) and \(I_2\) with \(\min I_2 - \max I_1\) sufficiently large and \(|A| \geq 3\), then
\[|A + A| = (2|I_1| - 1) + (|I_1| + |I_2| - 1) + (2|I_2| - 1) = 3|A| - 3 \text{ with } |P \setminus A| \text{ unbounded. Alternatively, taking } B = J \text{ to be a single interval } J \text{ with } \min I_2 - \max I_1 \text{ sufficiently large and } |A| \geq 3 \text{ gives } |A + B| = (|I_1| + |J| - 1) + (|I_2| + |J| - 1) = |A| + 2|B| - 2 \text{ with } |P \setminus A| \text{ again unbounded.}

As the latter examples above show, there is no way to approximate the summands in a small sumset \(A + B \subseteq \mathbb{Z}\) by arithmetic progressions once \(|A + B|\) becomes too large. However, the problematic examples above are quite limited, being instead the union of two arithmetic progressions. This is a special case of more general framework related to Freiman’s Theorem. As the first example shows, the \(3k - 4\) Theorem is actually one of the few instances in which the constants in Freiman’s Theorem are known precisely, and for distinct summands as well. For a fuller discussion of the framework related to Freiman’s Theorem, see [15] [9] [11]. The goal of this paper is to extend the precise estimates of the \(3k - 4\) Theorem to certain 2-dimensional sumsets under a more generous threshold bound for \(|A + B|\).

If \(G\) is an abelian group and \(A, B \subseteq G\) are nonempty subsets, then we can usually translate the sets \(A\) and \(B\) in any way and not significantly alter the structure of \(A, B\) or \(A + B\). To simplify notation, we then translate \(A\) and \(B\) so that \(0 \in A \cap B\). In this framework, sumsets, like other categorical objects, have an associated notion of morphism. A map \(\psi : A + B \to G'\), where \(G'\) is another abelian group, is called a normalized Freiman homomorphism if \(\psi(x + y) = \psi(x) + \psi(y)\) for all \(x \in A\) and \(y \in B\) (which implies \(\psi(0) = 0\)). Note \(0 \in A \cap B\) ensures that \(A, B \subseteq A + B\) are in the domain of \(\psi\), so this definition makes sense (in general, the key requirement is that there is a common element \(z \in A \cap B\) that can be used as a base point to the homomorphism, but it simplifies notation to further translate so that the common element is equal to \(z = 0\).

The image \(\psi(A + B) = \psi(A) + \psi(B)\) is the homomorphic image of \(A + B\), and the Freiman homomorphism \(\psi\) induces an isomorphism with its image, \(A + B \cong \psi(A) + \psi(B)\), when \(\psi\) is injective on all of \(A + B\) (not just on \(A\) and \(B\), which is in general too weak of a requirement).

With the notion of Fremin isomorphism in hand, it is possible to speak of the intrinsic dimension of a sumset \(A + B \subseteq G\), where \(0 \in A \cap B\), independent of the group \(G\) in which \(A + B\) is embedded. We define \(\dim^+(A + B)\) to be the maximal integer \(d \geq 0\) such that there is an injective normalized Freiman homomorphism \(\psi : A + B \to G'\) with \(\langle \psi(A) + \psi(B) \rangle = G'\) and \(G'\) having torsion-free rank \(\text{rk}(G') = d\). This dimension is independent of the translates of \(A\) and \(B\) chosen and, moreover, when the initial group \(G\) is torsion-free, there is an injective normalized Freiman homomorphism \(\psi : A + B \to \mathbb{Z}^d\) with \(\langle \psi(A) + \psi(B) \rangle = \mathbb{Z}^d\). See [6, Chapter 20] for details (derived via universal ambient groups). By a modification of the argument of [6, Proposition 3.1], one also deduces that there exist injective normalized Freiman homomorphisms \(\psi : A + B \to \mathbb{Z}^{d'}\) with \(\langle \psi(A) + \psi(B) \rangle = \mathbb{Z}^{d'}\) for any \(d' \in [1, d]\) as well.

When \(A + B\) is a torsion-free sumset (where \(A\) and \(B\) are finite and nonempty), meaning it has an embedding into a torsion free group, a result of Ruzsa [10] (combined with the remarks of the previous paragraph) shows that large dimension implies large sumset in the following sense:
dim⁺(A + B) ≥ d with |A| ≥ |B| implies

$$|A + B| ≥ |A| + d|B| - \frac{1}{2}d(d + 1).$$

For refinements to this result, see [15, Section 5.2] [7]. In particular, we see that that any finite, nonempty subset $A + B \subseteq \mathbb{Z}$ with dim⁺(A + B) ≥ 2 and |A| ≥ |B| has |A + B| ≥ |A| + 2|B| − 3, which corresponds roughly to the point after which the 3k − 4 Theorem breaks down. However, dim⁺(A + B) ≥ 3 and |A| ≥ |B| gives a bound of |A + B| ≥ |A| + 3|B| − 6, meaning the only obstacle to extending the 3k − 4 Theorem upwards towards the threshold |A| + 3|B| − 7 are 1- and 2-dimensional sumsets. For 2-dimensional sumsets, there is a refinement [7] of the result of Ruzsa.

**Theorem B.** Let $s ≥ 2$ be an integer. Let $A, B \subseteq \mathbb{R}^2$ be finite subsets with $|A| ≥ |B| ≥ 2s^2 − 3s + 2$. If

$$|A + B| < |A| + (3 - \frac{2}{s})|B| - 2s + 1,$$

then there is a line $\ell$ such that each of $A$ and $B$ can be covered by at most $s − 1$ parallel translates of $\ell$.

In particular (taking $s = 3$), when $|A| ≥ |B| ≥ 11$ and $|A + B| < |A| + \frac{7}{3}|B| - 5$, both $A$ and $B$ are covered by at most 2 parallel lines, meaning the only obstacles to extending the 3k − 4 Theorem upwards towards the threshold $|A| + \frac{7}{3}|B| - 5$ are 1-dimensional sumsets as well as 2-dimensional sumsets with both summands covered by at most 2 parallel lines. The latter sets are the subject of this paper. Indeed, our main results are the following, which extend the 3k − 4 Theorem to 2-dimensional sumsets with both summands covered by at most 2 parallel lines. Note we have normalized the hypotheses below so that these lines run parallel to $e_1$ with $A + B$ generating $\mathbb{Z}^2$ affinely. Also, $|\pi_2(A)|$ simply counts the number of lines parallel to $\mathbb{Z}e_1$ that are needed to cover $A$.

**Theorem 1.1.** Let $A, B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets with $|\pi_2(B)| = 1$ and $|\pi_2(A)| = 2$. Suppose $(A + B - A - B) = \mathbb{Z}^2$ and

$$|A + B| = |A| + 2|B| - 2 + r ≤ |A| + 2|B| + \min\{|A| - 1, |B|\} - 5.$$

Then there exist subsets $P_A, P_B \subseteq \mathbb{Z}^2$, with $P_B$ an arithmetic progression with difference $e_1$, and $P_A$ the union of two arithmetic progressions with difference $e_1$, such that $B \subseteq P_B$, $A \subseteq P_A$,

$$|P_A \setminus A| ≤ r \quad \text{and} \quad |P_B \setminus B| ≤ r.$$

**Theorem 1.2.** Let $A, B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets with $|\pi_2(A)| = |\pi_2(B)| = 2$. Suppose $(A + B - A - B) = \mathbb{Z}^2$, $|A| ≥ |B|$ and

$$|A + B| = |A| + 2|B| - 2 + r - \delta(A, B) ≤ 2|A| + 2|B| - 6 - \delta(A, B).$$
Then there exist subsets $P_A, P_B \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference $\epsilon_1$, such that $A \subseteq P_A$, $B \subseteq P_B$,

$$|P_A \setminus A| \leq r \quad \text{and} \quad |P_B \setminus B| \leq r.$$ 

**Theorem 1.3.** Let $A, B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets with $|\pi_2(A)| = |\pi_2(B)| = 2$. Suppose

$$\langle A + B - A - B \rangle = \mathbb{Z}^2, \quad |A| \geq |B|, \quad |A + B| \leq |A| + \frac{19}{2}|B| - 5 \quad \text{and} \quad |A + B| = |A| + 2|B| - 2 - \delta(A, B) + r = |A| + |B| + \frac{|A| + |B|}{2} - 3 + r'.$$

Then there exist subsets $P_A, P_B, P \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference $\epsilon_1$, such that $A \subseteq P_A$, $B \subseteq P_B$, $(x + A) \cup (y + B) \subseteq P$ for some $x, y \in \mathbb{Z}^2$,

$$|P_A \setminus A| \leq r, \quad |P_B \setminus B| \leq r, \quad |P_A \setminus A| + |P_B \setminus B| \leq 2r', \quad \text{and} \quad |P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r + 2 + \left|P_A \setminus P_B\right| - \left|A - B\right| \leq 3r + 2.$$

Moreover, $|P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r'$ unless

$$y + P_B \subseteq x + P_A = P \quad \text{and} \quad |P \setminus (x + A)| + |P \setminus (y + B)| = 2|P_A \setminus A| + |A| - |B|.$$

As with the $3k - 4$ Theorem, the bounds in the above theorems are precise (we will give examples later in the paper). Theorem 1.3 generalizes the two-dimensional case of [13] [14], which handles the symmetric case when $A = B$. We use many of the same compression methods of [12] [14] to reduce the 2-dimensional subset in consideration to several 1-dimensional subsets that can then be dealt with via the $3k - 4$ Theorem. However, unlike the case when $A = B$, much of the added difficulty in the proof of Theorem 1.3 will arise from trying to show $A$ and $B$ can be approximated simultaneously by the same progression $P$, a fact which holds trivially when $A = B$, and which clearly cannot hold in Theorem 1.1.

**2. Refined Theory Involving the 3k - 4 Theorem**

There are some important consequences of Theorem A that “refine” its use in practice. These details are all contained in [1], but we repeat them here owing to their critical role in the proof of Theorem 1.3. Given a set $X \subseteq \mathbb{Z}$, we let $P_X \subseteq \mathbb{Z}$ denote the minimal arithmetic progression with difference 1 containing $X$. Let $A, B \subseteq \mathbb{Z}$ be finite, nonempty subsets with $\langle A + B - A - B \rangle = \mathbb{Z}^2$,

$$\text{diam}(B) \leq \text{diam}(A) \leq |A| + |B| - 3 \quad \text{and} \quad |A + B| \leq |A| + 2|B| - 3 - \delta(A, B). \quad (2)$$

Moreover, let $|A + B| = |A| + |B| - 1 + r$. These assumptions will apply throughout the discussion of this section.

The quantity $|P_A \setminus A|$ counts the number of “holes” in $A$ with respect to $P_A$, and it is easily noted that $\text{diam} = |A| + |P_A \setminus A| - 1$. There are various ways to obtain the hypothesis

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**5 Theorem.** However, unlike the case when

$$|A + B| = |A| + 2|B| - 2 - \delta(A, B) + r = |A| + |B| + \frac{|A| + |B|}{2} - 3 + r'.$$

Then there exist subsets $P_A, P_B, P \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference $\epsilon_1$, such that $A \subseteq P_A$, $B \subseteq P_B$, $(x + A) \cup (y + B) \subseteq P$ for some $x, y \in \mathbb{Z}^2$,

$$|P_A \setminus A| \leq r, \quad |P_B \setminus B| \leq r, \quad |P_A \setminus A| + |P_B \setminus B| \leq 2r', \quad \text{and} \quad |P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r + 2 + \left|P_A \setminus P_B\right| - \left|A - B\right| \leq 3r + 2.$$

Moreover, $|P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r'$ unless

$$y + P_B \subseteq x + P_A = P \quad \text{and} \quad |P \setminus (x + A)| + |P \setminus (y + B)| = 2|P_A \setminus A| + |A| - |B|.$$
the above being a useful and equivalent reformulation of (1). Combined with the upper bound
\[ \text{diam} A \leq |A| + |B| - 3, \] which is equivalent to \( |P_A \setminus A| \leq |B| - 2 \) in view of the previous equality.
For instance, if
\[ \gcd(A - A) \leq 2, \quad \text{diam}(B) \leq \text{diam}(A) \quad \text{and} \quad |A + B| \leq |A| + 2|B| - 3 - \delta(A, B), \] then Theorem A(ii) implies that
\[ |A + B| \geq |A| + |B| - 1 + \max\{|P_A \setminus A|, |P_B \setminus B|\}, \] the above being a useful and equivalent reformulation of (1). Combined with the upper bound for \( |A + B| \) from (3), it then follows that \( |P_A \setminus A| \leq |B| - 2 - \delta(A, B) \). In summary, the hypotheses from (3) imply those of (2) via Theorem A and will be the usual way that we obtain the setup found in (2).

Regardless, assuming \( A \) and \( B \) satisfy (2), we have the existence of an arithmetic progression \( R \subseteq \mathbb{Z} \) with difference 1 such that
\[ R \subseteq A + B \quad \text{and} \quad |R| \geq |A| + |B| - 1 \]
per the main result of [1]. We also have \( R \subseteq P_{A+B} \) with \( |P_{A+B}| = |A| + |B| - 1 + |P_A \setminus A| + |P_B \setminus B| \), meaning \( R \) covers all but \( |P_A \setminus A| + |P_B \setminus B| \) elements of the interval \( P_{A+B} \). Assuming (4) holds (as would be the case under the assumptions of (3)), we find that
\[ |P_{A+B} \setminus (A + B)| \leq \min\{|P_A \setminus A|, |P_B \setminus B|\}. \]

The existence of the interval \( R \subseteq A + B \) with \( |R| \geq |A| + |B| - 1 \) has several important consequences, all derived in [1]. First,
\[ \min P_{A+B} + \left[ |P_A \setminus A| + |P_B \setminus B|, |A| + |B| - 2 \right] \subseteq A + B, \]
for there are simply not enough elements in \( P_{A+B} \) for the arithmetic progression \( R \) to avoid this interval no matter where \( R \subseteq P_{A+B} \) occurs. Since \( \min A + \max B - 1, \min B + \max A + 1 \) is contained in the above interval (given the assumptions of (2)), a particular consequence is that
\[ \min A + \max B - 1, \min B + \max A + 1 \subseteq A + B. \]
Thus all elements of \( P_{A+B} \setminus (A + B) \) are contained in \( P_{\min A+B} \cup P_{\max A+B} \). In particular, if \( x \in P_{A+B} \setminus (A + B) \), then either
\[ - \min A + x \in P_B \setminus B \quad \text{or} \quad - \max A + x \in P_B \setminus B. \]
An element \( x \in P_{A+B} \setminus (A + B) \) of the first type is called a left hole in \( A + B \), and those of the second type are called right holes. Equivalently, since \( \min A + \max B - 1, \min B + \max A + 1 \subseteq R \subseteq A + B \), it follows that a hole \( x \in P_{A+B} \setminus (A + B) \) is left if \( x < \min R \) and is right if \( x > \max R \), which means that if \( x \) is a left hole and \( y \) is a right hole, then
\[ y - x \geq (\max R + 1) - (\min R - 1) \geq |A| + |B|. \]

Likewise, an element \( x \in P_B \setminus B \) with \( \min A + x \in P_{A+B} \setminus (A + B) \) is called a left stable hole in \( B \), an element \( x \in P_B \setminus B \) with \( \max A + x \in P_{A+B} \setminus (A + B) \) is called a right stable hole in \( B \), and all
other \( x \in P_B \setminus B \) are called unstable holes. Observing that \((\max A + x) - (\min A + x) = \text{diam} A \leq |A| + |B| - 3\), we conclude that no stable hole in \( B \) can be both left and right stable. Moreover, if \( x \in P_B \setminus B \) is left stable and \( y \in P_B \setminus B \) is right stable, then \((y + \max A) - (x + \min A) \geq |A| + |B|\), which implies \( y - x \geq |B| - |P_A \setminus A| + 1 \geq 3 \). Thus all left stable holes precede all right stable holes in \( B \). Similar definitions and conclusions hold regarding \( A \): an element \( x \in P_A \setminus A \) with \( \min B + x \in P_{A+B} \setminus (A + B) \) is called a left stable hole in \( A \), an element \( x \in P_A \setminus A \) with \( \max B + x \in P_{A+B} \setminus (A + B) \) is right stable, and all left stable holes precede right stable ones in \( A \).

Let \( e \) be the greatest left stable hole in \( B \) (or \( \min B - 1 \) if none exist) and let \( c \) be the smallest right stable hole in \( B \) (or \( \max B + 1 \) if none exist). Then \( J_B = [e + 1, c - 1] \subseteq P_B \) is a nonempty interval. Moreover, \( A + J_B \subseteq P_A + J_B = [\min A + e + 1, \max A + c - 1] \subseteq A + B \) per definition of left and right holes and the extremal assumptions on \( e \) and \( c \), which means \( A + (B \cup J_B) = A + B \). Noticing that the left stable hole \( x \in P_B \setminus B \) corresponds to the left hole \( \min A + x \in P_{A+B} \setminus (A + B) \) and then to the left stable hole \( \min A - \min B + x \in P_A \setminus A \), with similar statements holding for right holes, we find that \( J_A = [\min A - \min B + e + 1, \max A - \max B + c - 1] \subseteq P_A \) has \((A \cup J_A) + (B \cup J_B) = A + B \). This has important consequences. For instance, if we were applying the \( 3k - 4 \) Theorem to \( A + B \), then we can instead imply it to \((A \cup J_A) + (B \cup J_B)\) resulting in better bounds. Indeed, (4) then improves by one for each element of \( J_A \setminus A \) and \( J_B \setminus B \):

\[
|A + B| \geq |A| + |B| - 1 + \max\{|P_A \setminus A| + |J_B \setminus B|, |P_B \setminus B| + |J_A \setminus A|\}. \tag{5}
\]

Likewise the bound on the size of \( |R| \) increases to

\[
|R| \geq |A| + |B| - 1 + |J_B \setminus B| + |J_A \setminus A|,
\]

with corresponding improvements in other associated bounds mentioned above.

By definition of \( e \) and \( c \), we have \( \min A + e \notin A + B \) and \( \max A + c \notin A + B \). Thus the progression \( R \) must lie wholly in one of the intervals \([\min A + \min B, \min A + e - 1], [\max A + c + 1, \max A + \max B] \) or \([\min A + e + 1, \max A + c - 1] \). Since the first two interval have respective sizes \( e - \min B < \max B - \min B = \text{diam} B \leq |A| + |B| - 3 < |R| \) and \( \max B - c < \max B - \min B \leq |A| + |B| - 3 < |R| \), we conclude that \( R \subseteq [\min A + e + 1, \max A + c - 1] \), implying that \( \text{diam} A - 1 + c - e = |A| + |P_A \setminus A| - 1 + |J_B| \geq |R| \geq |A| + |B| - 1 + |J_B \setminus B| + |J_A \setminus A| \).

Thus

\[
|J_B| \geq |B| - |P_A \setminus A| + |J_A \setminus A| + |J_B \setminus B|,
\]

and

\[
|J_A| = |J_B| + \text{diam} A - \text{diam} B \geq |A| - |P_B \setminus B| + |J_A \setminus A| + |J_B \setminus B|.
\]

3. Setup and Proof of Theorems 1.1, 1.2 and 1.3

We will frequently use the following basic and easily proven bound for torsion free sumsets (see [6, Theorem 3.1]).
Theorem C. Let $G$ be a torsion free abelian group and let $A, B \subseteq G$ be finite and nonempty subsets. Then

$$|A + B| \geq |A| + |B| - 1.$$ 

A simple corollary of the above result is the following.

Corollary 3.1. Let $A, B \subseteq \mathbb{Z}$ be finite, nonempty subsets and let $d \geq 1$. Suppose $d \mid \gcd(B - B)$ and let $s \in [1, d]$ denote the number of $d\mathbb{Z}$-cosets that intersect $A$. Then

$$|A + B| \geq |A| + s(|B| - 1).$$

In particular, if $\gcd(B - B) \neq 1$ and $\gcd(A - A) = 1$, then $|A + B| \geq |A| + 2|B| - 2$.

Proof. Let $x_1, \ldots, x_s \in A$ be a set of representatives for the $d\mathbb{Z}$-cosets that intersect $A$. These cosets give rise to a partition $A = \bigcup_{i=1}^s A_i$, where $A_i = (x_i + d\mathbb{Z}) \cap A \neq \emptyset$. The hypothesis $d \mid \gcd(B - B)$ ensures that $B$ is itself contained in a single $d\mathbb{Z}$-coset. Thus the $A_i + B$ are each contained in distinct $d\mathbb{Z}$-cosets, implying $|A + B| = \sum_{i=1}^s |A_i + B|$. Applying Theorem C to each $|A_i + B|$ yields the desired bound. Finally, if $d = \gcd(B - B) > 1$ and $\gcd(A - A) = 1$, then $s \geq 2$ follows, and now the previous bound implies $|A + B| \geq |A| + 2|B| - 2$. This bound is trivial when $|B| = 1$, i.e., when $\gcd(B - B) = 0$. \hfill \Box

We next introduce several notions for 2-dimensional sets. The first is that of compression, a concept that has been exploited to much success in additive theory [2] [4] [5] [6, Chapter 7] [7] [14].

Let $A \subseteq \mathbb{Z}^2$ be a finite subset. The linear compression of $A$, with respect to $e_2$ and denoted $C_{e_2}(A)$, is the set obtained by compressing and shifting $A$ along each vertical line until the resulting set is an arithmetic progression with difference $e_2$ whose first term is contained on the horizontal axis. More concretely, we define the set $C_{e_2}(A) \subseteq \mathbb{Z}^2$ piecewise by it’s intersections with the lines $Ze_2 + xe_1$, for $x \in \mathbb{Z}$, by letting $C_{e_2}(A) \cap (Ze_2 + xe_1)$ be the subset of $Ze_2 + xe_1$ satisfying

$$\pi_2(C_{e_2}(A) \cap (Ze_2 + xe_1)) = \{0, 1, 2, \ldots, (r-1)\},$$

where $|A \cap (Ze_2 + xe_1)| = r$ and the right hand side is considered empty if $r = 0$. Letting $C_t := C \cap (Ze_2 + te_1)$ below, it follows in view of Theorem C that

$$|A + B| = \sum_{t \in \mathbb{Z}} |(A + B)_t|$$

$$\geq \sum_{t \in \mathbb{Z}} \max\{|A_s + B_{t-s}| : \ A_s \neq \emptyset, B_{t-s} \neq \emptyset\}$$

$$\geq \sum_{t \in \mathbb{Z}} \max\{|A_s| + |B_{t-s}| - 1 : \ A_s \neq \emptyset, B_{t-s} \neq \emptyset\}$$

$$= |C_{e_2}(A) + C_{e_2}(B)|,$$ (6)
for finite subsets $A, B \subseteq \mathbb{Z}^2$.

For Theorems 1.1, 1.2 and 1.3, we have $|\pi_2(A)| = 2$ and $|\pi_2(B)| \leq 2$. By translating appropriately, we may assume

\[ A = A_0 \cup A_1 \quad \text{and} \quad B = B_0 \cup B_1 \]

with $A_0, B_0 \subseteq \mathbb{Z}e_1$ both nonempty and $A_1$ and $B_1$ the elements of $A$ and $B$, respectively, not contained on the horizontal line $\mathbb{Z}e_1$. Observe this means all elements of $A_1$ lie on a horizontal line parallel to $\mathbb{Z}e_1$, say on $\mathbb{Z}e_1 + xe_2$, as do all elements of $B_1$, say on $\mathbb{Z}e_1 + ye_2$. Thus $B_1$ is nonempty precisely when $|\pi_2(B)| = 2$ rather than $|\pi_2(B)| = 1$. If $|\pi_2(B)| = 2$ and $|x| \neq |y|$, then applying Theorem C four times gives $\mathbb{Z}$ line parallel to $\mathbb{Z}$, \( \mathbb{Z} \) nonempty precisely when $P_\alpha$ amount $(\mathbb{A}$ has the effect of shifting both the sets $\mathbb{A}$ in (7). Indeed, to see this, simply choose a horizontal shift such that equality holds in the first inequality of (7). Thus, leaving $A_0$ and $B_0$ fixed but translating $A_1$ and $B_1$ by the same constant $\alpha e_1$, where $\alpha \in \mathbb{Z}$, has the effect of shifting both the sets $A_1$ and $B_1$ along the horizontal line $\mathbb{Z}e_1 + xe_2$ by the amount $\alpha$. This is the same as applying the linear transformation $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $(x, y) \mapsto (x + \alpha y, y)$, which is easily seen to have determinant 1 and thus map $\mathbb{Z}^2$ isomorphically onto $\mathbb{Z}^2$. We will call such a linear transformation $\psi$ a horizontal shift. Using a horizontal shift by $\alpha$, we can replace the sets $A_1$ and $B_1$ by $A_1 + \alpha e_1$ and $B_1 + \alpha e_1$, in effect, allowing us to slide the sets $A_1$ and $B_1$ by the same amount along the line $\mathbb{Z}e_1 + e_2$. Since it is readily seen that the desired conclusions holding for the sumset $\psi(A) + \psi(B)$ imply that the desired conclusions hold for the original sumset $A + B$, we will make free use of horizontal shifts in the proofs below.

For instance, given any fixed $x \in \{0, 1\}$, it is easily seen that there is a horizontal shift $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ so that

\[
\max \pi_1(\psi(A_x)) \leq \min \pi_1(\psi(A_{1-x})) \quad \text{and} \quad \max \pi_1(\psi(B_x)) \leq \min \pi_1(\psi(B_{1-x})),
\]  

(7)

where $\max \emptyset := +\infty$ and $\min \emptyset := -\infty$, with equality holding in at least one of the estimates in (7). Indeed, to see this, simply choose a horizontal shift such that equality holds in the first inequality of (7). Then, if the second inequality in (7) fails, further shift the sets $A_{1-x}$ and $B_{1-x}$ to the right until equality holds in the second inequality of (7). As further shifting $A_{1-x}$ only increases $\min \pi_1(\psi(A_{1-x}))$, the first inequality of (7) remains true, showing (7) holds.
Let \( \tilde{A} := C_{w_2}(\psi(A)) \) and \( \tilde{B} := C_{w_2}(\psi(B)) \), with \( \tilde{A} = A_0 \cup \tilde{A}_1 \) and \( \tilde{B} = B_0 \cup \tilde{B}_1 \), where \( \tilde{A}_i = A \cap (Ze_1 + ie_2) \) and \( \tilde{B}_i = B \cap (Ze_1 + ie_2) \) for \( i \in \{0, 1\} \). Since \( \langle A + B - A - B \rangle = \mathbb{Z}^2 \) with \( \psi : \mathbb{Z}^2 \to \mathbb{Z}^2 \) an isomorphism, it follows that

\[
\langle \psi(A + B - A - B) \rangle = \mathbb{Z}^2.
\]

Moreover, we have

\[
\langle \tilde{A}_0 + \tilde{B}_0 - \tilde{A}_0 - \tilde{B}_0 \rangle = \mathbb{Z}e_1,
\]

for if instead \( \langle \tilde{A}_0 + \tilde{B}_0 - \tilde{A}_0 - \tilde{B}_0 \rangle = d\mathbb{Z}e_1 \) with \( d \geq 2 \), then it is easily seen that \( \langle \psi(A) - \psi(A) \rangle \leq \langle \tilde{A}_0 - \tilde{A}_0 \rangle \times \mathbb{Z} \leq \langle \tilde{A}_0 + \tilde{B}_0 - \tilde{A}_0 - \tilde{B}_0 \rangle \times \mathbb{Z} \) (the first inclusion follows in view of \( |\tilde{A}_1| \leq 1 \); see (10) below) and \( \langle \psi(B) - \psi(B) \rangle \leq \langle \tilde{B}_0 - \tilde{B}_0 \rangle \times \mathbb{Z} \leq \langle \tilde{A}_0 + \tilde{B}_0 - \tilde{A}_0 - \tilde{B}_0 \rangle \times \mathbb{Z} = \mathbb{Z}e_1 \times \mathbb{Z} \) (the first inclusion follows in view of \( |\tilde{B}_1| \leq 1 \)), contradicting that \( \langle \psi(A) - \psi(A) \rangle + \langle \psi(B) - \psi(B) \rangle = \langle \psi(A) + \psi(B) - \psi(A) - \psi(B) \rangle = \mathbb{Z}^2 \). The following properties are also easily observed:

\[
|\tilde{A}| = |A| \quad \text{and} \quad |\tilde{B}| = |B|, \quad \text{(9)}
\]

\[
|\tilde{A}_1|, |\tilde{B}_1| \leq 1 \quad \text{and} \quad \max\{|\tilde{A}_1|, |\tilde{B}_1|\} = 1, \quad \text{(10)}
\]

\[
|P_A \setminus \tilde{A}| = |P_{\tilde{A}_0} \setminus \tilde{A}_0| \quad \text{and} \quad |P_B \setminus \tilde{B}| = |P_{\tilde{B}_0} \setminus \tilde{B}_0|, \quad \text{(11)}
\]

\[
|A + B| \geq |\tilde{A} + \tilde{B}|, \quad \text{(13)}
\]

where (13) follows in view of (6). The use of such pairs \((\tilde{A}, \tilde{B})\) to study more general configurations of points in the plane follows that of Freiman [4] and Stanchescu [14]. They will allow us to attain the desired bounds for \( |P_A \setminus A| \) and \( |P_B \setminus B| \) rather easily.

**Proof of Theorem 1.1.** Let \( \tilde{A} = A_0 \cup \tilde{A}_1 \) and \( \tilde{B} = B_0 \cup \tilde{B}_1 \) be as defined above. Then, since \( B_1 = \emptyset \), it follows that \( \tilde{B}_1 = 0, \tilde{B}_0 = B, |\tilde{A}_1| = 1 \) and \( |\tilde{A}_0| = |\tilde{A}| - 1 \). Consequently, it follows from (9) and (13) that

\[
|\tilde{A}_0 + \tilde{B}| + |\tilde{B}| = |\tilde{A} + \tilde{B}| \leq |A + B| = |\tilde{A}_0| + 2|\tilde{B}| - 1 + r \leq |\tilde{A}_0| + 2|\tilde{B}| + \min\{|\tilde{A}_0|, |\tilde{B}|\} - 4.
\]

Thus, in view of (8), we can apply Theorem A(i) (with \( d = 1 \)) to \( \tilde{A}_0 + \tilde{B} \) to conclude \( |P_{\tilde{A}_0} \setminus \tilde{A}_0| \leq r \) and \( |P_{\tilde{B}} \setminus \tilde{B}| \leq r \), and now the proof is complete in view of (12) and (11). \( \square \)

**Proof of Theorem 1.2.** Let \( \tilde{A} = A_0 \cup \tilde{A}_1 \) and \( \tilde{B} = B_0 \cup \tilde{B}_1 \) be as defined above. From our hypotheses and (13) and (9), we find that

\[
|\tilde{A} + \tilde{B}| \leq |A + B| \leq |\tilde{A}| + 2|\tilde{B}| - 2 + r \leq |\tilde{B}| + 2|\tilde{A}| - 2 + r
\]

and

\[
|\tilde{A} + \tilde{B}| \leq |A + B| \leq 2|\tilde{A}| + 2|\tilde{B}| - 6.
\]

Thus, if either \( \tilde{B}_1 = 0 \) or \( \tilde{A}_1 = 0 \), we can apply Theorem 1.1 to \( \tilde{A} + \tilde{B} \) to conclude \( |P_{\tilde{A}} \setminus \tilde{A}| \leq r \) and \( |P_{\tilde{B}} \setminus \tilde{B}| \leq r \), in which case the proof is complete in view of (12). Therefore, in view of (10),
and by appropriately translating the sets $A$ and $B$, we may instead assume $\tilde{A}_1 = \tilde{B}_1 = \{0\} \times \{1\}$, in which case

$$|\tilde{A}_0| = |A| - 1 \geq |B| - 1 = |\tilde{B}_0|.$$ 

Note

$$|A + B| \geq |\tilde{A} + \tilde{B}| = |\tilde{A}_0 + \tilde{B}_0| + |(0 + \tilde{B}_0) \cup (\tilde{A}_0 + 0)| + 1 = |\tilde{A}_0 + \tilde{B}_0| + |\tilde{B}_0| + |\tilde{A}_0| - |\tilde{B}_0 \cap \tilde{A}_0| + 1. \quad (14)$$

We have the trivial estimate

$$|\tilde{B}_0 \cap \tilde{A}_0| \leq |\tilde{B}_0|, \quad (15)$$

with equality only possible if $\tilde{B}_0 \subseteq \tilde{A}_0$. We also have the trivial estimate

$$|\tilde{B}_0 \cap \tilde{A}_0| \leq |\tilde{A}_0| \quad (16)$$

with equality only possible if $\tilde{A}_0 = \tilde{B}_0$ (in view of $|\tilde{A}_0| \geq |\tilde{B}_0|$). However, in view of our normalization assumption $\tilde{A}_1 = \tilde{B}_1 = \{0\} \times \{1\}$ and the definition of $\tilde{A}$ and $\tilde{B}$, it follows that the equality $\tilde{A}_0 = \tilde{B}_0$ is only possible if $A = B$. In particular, it is only possible if $\delta(A, B) = 1$, allowing us to refine the estimate in (16) to

$$|\tilde{B}_0 \cap \tilde{A}_0| \leq |\tilde{A}_0| - 1 + \delta(A, B). \quad (17)$$

By hypothesis, we have

$$|A + B| = |\tilde{A}| + 2|\tilde{B}| - 2 + r - \delta(A, B) = |\tilde{A}_0| + 2|\tilde{B}_0| + 1 + r - \delta(A, B) \quad \text{and} \quad (18)$$

$$|A + B| \leq 2|\tilde{A}| + 2|\tilde{B}| - 6 - \delta(A, B) = 2|\tilde{A}_0| + 2|\tilde{B}_0| - 2 - \delta(A, B). \quad (19)$$

Combining (14), (17) and (18) yields

$$|\tilde{A}_0 + \tilde{B}_0| \leq |\tilde{A}_0| + |\tilde{B}_0| - 1 + r.$$

Combining (14), (15) and (19) yields

$$|\tilde{A}_0 + \tilde{B}_0| \leq |\tilde{A}_0| + 2|\tilde{B}_0| - 3 - \delta(A, B),$$

with strict inequality unless $\tilde{B}_0 \subseteq \tilde{A}_0$. If we have have strict inequality, then

$$|\tilde{A}_0 + \tilde{B}_0| \leq |\tilde{A}_0| + 2|\tilde{B}_0| - 3 - \delta(\tilde{A}_0, \tilde{B}_0). \quad (20)$$

On the other hand, if $\tilde{B}_0 \subseteq \tilde{A}_0$, then $\delta(\tilde{A}_0, \tilde{B}_0) = 1$ (which implies $|\tilde{A}_0| \leq |\tilde{B}_0|$) is only possible if $\tilde{A}_0 = \tilde{B}_0$, further implying $A = B$ and $\delta(A, B) = 1$. Thus $\delta(A, B) \geq \delta(\tilde{A}_0, \tilde{B}_0)$, so that (20) holds in this case as well. Consequently, in view of (8), we can apply Theorem A(i) (with $d = 1$) to $\tilde{A}_0 + \tilde{B}_0$ to conclude $|P_{\tilde{A}_0 \setminus \tilde{A}_0}| \leq r$ and $|P_{\tilde{B}_0 \setminus \tilde{B}_0}| \leq r$, and now the proof is complete in view of (12) and (11).

We now come to the proof of our main theorem.
Proof of Theorem 1.3. The assumption $|A| \geq |B|$ is present in the statement of Theorem 1.3 solely to simplify its presentation. However, to take advantage of symmetry in what follows, we will not assume this in the proof. Instead, let $\tilde{A}$ be a set from among $A$ and $B$ with larger cardinality and let $\tilde{B}$ be the other set. By hypothesis,
\begin{equation}
|A + B| \leq |\tilde{A}| + \frac{19}{4}|	ilde{B}| - 5,
\end{equation}
which implies
\begin{equation}
|A + B| \leq |\tilde{A}| + 3|\tilde{B}| - 6 \quad \text{and} \quad |A + B| \leq 2|\tilde{A}| + 2|\tilde{B}| - 6 - \delta(\tilde{A}, \tilde{B})
\end{equation}
except when $\delta(A,B) = \delta(B,A) = 1$ with $|A| = |B| \leq 3$. However, in view $\langle A + B - A - B \rangle = \mathbb{Z}^2$, this is only possible if we can translate $A$ and $B$ so that $A = B$, $|A| = |B| = 3$ and $|A + B| = 3 + 2 + 1 = 6 = |A| + |B| - 2 - \delta(A,B) = |A| + |B| + |A| |B| - 3$, implying $r = r' = 0$. In such case, the theorem holds using $A = B = P = P_A = P_B$. As a result, we can assume (22) holds. Hence we can apply Theorem 1.2 to conclude that
\begin{equation}
|P_A \setminus A| \leq r \quad \text{and} \quad |P_B \setminus B| \leq r.
\end{equation}
In particular, $r \geq 0$. We must show $|P_A \setminus A| + |P_B \setminus B| \leq 2r'$ and
\begin{equation}
|P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r + 2 + |P_A| - |P_B| - \left|A| - |B|\right|
\end{equation}
for some $x, y \in \mathbb{Z}^2$ with $P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r'$ unless either $P_B \subseteq P_A$ or $P_A \subseteq P_B$.

Observe that $|A| + |P_A \setminus A| = |P_A|$ and $|B| + |P_B \setminus B| = |P_B|$. Thus, if $|P_A| \geq |P_B|$, then
\begin{equation}
\left|P_A| - |P_B|\right| - \left|A| - |B|\right| = |P_A \setminus A| - |P_B \setminus B| - 2(|B| - \min\{|A|, |B|\}),
\end{equation}
which is at most $r$ in view of (23). Likewise, if $|P_B| \geq |P_A|$, then $\left|P_A| - |P_B|\right| - \left|A| - |B|\right| = |P_B \setminus B| - |P_A \setminus A| - 2(|A| - \min\{|A|, |B|\})$, which is again at most $r$ by (23). In either case, the bound in (24) is at most $3r + 2$.

If $\delta(\tilde{A}, \tilde{B}) = 1$, then $\tilde{A} \geq |\tilde{B}|$ forces $x + \tilde{A} = \tilde{B}$ for some $x \in \mathbb{Z}^2$, whence, by an appropriate translation, we can assume $A = B$. But then $r' = r$ and $P = P_A = P_B$ trivially has
\begin{equation}
|P \setminus A| + |P \setminus B| = |P_A \setminus A| + |P_B \setminus B| \leq 2r = 2r',
\end{equation}
as desired. Therefore we can assume
\begin{equation}
\delta(\tilde{A}, \tilde{B}) = 0, \quad r = r' + \frac{1}{2}|A| - |B| - 1, \quad \text{and} \quad 2r + 2 - |A| - |B| = 2r'.
\end{equation}
By definition of $r$, $\tilde{A}$ and $\tilde{B}$, we have $|A + B| = |\tilde{A}| + 2|\tilde{B}| - 2 + r + \delta(\tilde{A}, \tilde{B})$. Thus (22) implies
\begin{equation}
r \leq |\tilde{B}| - \delta(\tilde{A}, \tilde{B}) - 4 = |\tilde{B}| - 4 = \min\{|A|, |B|\} - 4.
\end{equation}
We trivially have $|P \setminus (x + A)| \geq |P_A \setminus A|$ and $|P \setminus (y + B)| \geq |P_B \setminus B|$. Thus the inequality $|P \setminus (x + A)| + |P \setminus (y + B)| \leq 2r + 2 - |A| - |B| = 2r'$ implies $|P_A \setminus A| + |P_B \setminus B| \leq 2r'$, meaning we only need to separately verify $|P_A \setminus A| + |P_B \setminus B| \leq 2r'$ when the former bound fails.
To simplify notation, whenever we consider a subset \( X \) contained in a horizontal line, we will use notation and language from \( \mathbb{Z} \), such as \( \text{diam} X \), \(<\), \( \text{gcd}(X) \), an interval in \( X \), etc., to refer to the corresponding concept for \( \pi_1(X) \). For example, an interval \( I \subseteq X \) is a set such that \( \pi_1(I) \) is an interval in \( \mathbb{Z} e_1 \), \( \text{diam} X = \max \pi_1(X) - \min \pi_1(X) \), and \( x < y \), for elements \( x, y \in X \), means \( \pi_1(x) < \pi_1(y) \), etc. Likewise \( \max X \in X \) is the element \( x \in X \) with \( \pi_1(x) = \max \pi_1(X) \), and \( \min X \in X \) is the element \( x \in X \) with \( \pi_1(x) = \min \pi_1(X) \), which both exist when \( X \) is finite and nonempty.

By exchanging the roles of \( A \) and \( B \) if need be and translating, we may w.l.o.g. assume \( P_{B_0} \subseteq P_{A_0} \). Then, by appropriately translating \( B \) and possibly applying the linear transformation \( (x, y) \mapsto (-x, y) \), we can further assume one of the following three cases holds:

A: \( P_{B_0} \subseteq P_{A_0} \) and \( P_{A_1} \subseteq P_{B_1} \),

B: \( P_{B_0} \subseteq P_{A_0} \) and \( P_{B_1} \subseteq P_{A_1} \),

C: \( P_{B_0} \subseteq P_{A_0} \), \( \min A_0 = \min B_0 \), \( \min A_1 < \min B_1 \), and \( \max A_1 < \max B_1 \).

With the translation assumptions above, let

\[
P = P_{A \cup B}.
\]

We handle the above three cases separately.

**Case C.** In this case, there is a horizontal shift \( \psi_1 \) with

\[
\pi_1\left(\max \psi_1(A_0)\right) = \pi_1\left(\min \psi_1(A_1)\right) \quad \text{and} \quad \pi_1\left(\max \psi_1(B_0)\right) < \pi_1\left(\min \psi_1(B_1)\right),
\]

and also an horizontal shift \( \psi_2 \) with

\[
\pi_1\left(\max \psi_2(B_1)\right) = \pi_1\left(\min \psi_2(B_0)\right) \quad \text{and} \quad \pi_1\left(\max \psi_2(A_1)\right) < \pi_1\left(\min \psi_2(A_0)\right).
\]

Let \( A' = C_{e_2}(\psi_1(A)) \), \( B' = C_{e_2}(\psi_1(B)) \), \( A'' = C_{e_2}(\psi_2(A)) \), and \( B'' = C_{e_2}(\psi_2(B)) \). From the above, we have \( B' = B'_0 \), \( A' = A'_0 \cup A'_1 \) with \( |A'| = 1 \), \( B'' = B''_0 \cup B''_1 \) with \( |B''_1| = 1 \), and \( A'' = A''_0 \). Moreover, \( (A' + B' - A' - B') = (A'' + B'' - A'' - B'') = \mathbb{Z}^2 \) by (8), and in view of the hypotheses of Case C, we find that

\[
|P_{A''} \setminus A''| + |P_{B'} \setminus B'| = (|P \setminus A| - 1) + (|P \setminus B| - 1).
\]

(26)

Now (13), (9) and (22) imply \( |A' + B'| \leq |A + B| = |A'| + |B'| + 2 \min\{|A'|, |B'|\} - 6 \). Likewise, \( |A'' + B''| \leq |A + B| \leq |A''| + |B''| + 2 \min\{|A''|, |B''|\} - 6 \). Also, (13), (9) and the hypotheses of Theorem 1.3 give

\[
|A' + B'| \leq |A + B| = |A| + |B| + \min\{|A|, |B|\} - 2 + r
= |A'| + 2|B'| - 2 + r - (|B| - \min\{|A|, |B|\})
\]

and, likewise, \( |A'' + B''| \leq |B''| + 2|A''| - 2 + r - (|A| - \min\{|A|, |B|\}) \). But this means we can apply Theorem 1.1 to \( A' + B' \) and \( A'' + B'' \) to conclude \( |P_{B'} \setminus B'| \leq r - (|B| - \min\{|A|, |B|\}) \).
and $|P_{A'} \setminus A''| \leq r - (|A| - \min\{|A|, |B|\})$. Combined with (26), we obtain the desired bound

$$|P \setminus A| + |P \setminus B| \leq 2r + 2 - |A| - |B| = 2r'.$$

**Case B.** In this case, $P_B \subseteq P_A$, implying $P = P_A$, so that we trivially have $|P \setminus A| + |P \setminus B| = 2|P_A| - |A| - |B| = 2|P_A \setminus A| + |A| - |B|$. Moreover, there is a horizontal shift $\psi_1$ with

$$\max \psi_1(A_0) = \pi_1\left(\min \psi_1(A_1)\right) \quad \text{and} \quad \pi_1(\max \psi_1(B_0)) \leq \pi_1\left(\min \psi_1(B_1)\right),$$

and also an horizontal shift $\psi_2$ with

$$\pi_1(\max \psi_2(A_1)) = \pi_1(\min \psi_2(A_0)) \quad \text{and} \quad \pi_1(\max \psi_2(B_1)) \leq \pi_1\left(\min \psi_2(B_0)\right).$$

Let $A' = C_{e_2}(\psi_1(A))$, $B' = C_{e_2}(\psi_1(B))$, $A'' = C_{e_2}(\psi_2(A))$, and $B'' = C_{e_2}(\psi_2(B))$ as in Case C. If both $\pi_1(\max \psi_1(B_0)) = \pi_1\left(\min \psi_1(B_1)\right)$ and $\pi_1(\max \psi_2(B_1)) = \pi_1\left(\min \psi_2(B_0)\right)$, then $P = P_A = P_B$ and $|A| + |P \setminus A| = |B| + |P \setminus B|$. Thus, if $|A| \geq |B|$, then (23) implies $|P \setminus A| = |P \setminus B| - (|A| - |B|) \leq r - (|A| - |B|)$. Likewise, if $|B| \geq |A|$, then (23) trivially gives $|P \setminus B| = |P \setminus A| - (|B| - |A|) \leq r - (|B| - |A|)$. In either case, we see that one of the two estimates in (23) can be improved by $|A| - |B|$, yielding the desired bound

$$|P \setminus A| + |P \setminus B| \leq 2r - |A| - |B| = 2r - 2.$$

Suppose equality holds in only one of the two, say $\pi_1\left(\max \psi_2(B_1)\right) = \pi_1\left(\min \psi_2(B_0)\right)$ (the other case is nearly identical using $A'' + B''$ in place of $A' + B'$). Then $B' = B_0$, $A' = A_0' \cup A_1'$ with $|A_1'| = 1$, $|P_{A'} \setminus A'| = |P_A \setminus A| = |P \setminus A|$ and $|P_{B'} \setminus B'| = |P_A \setminus B| - 1 = |P \setminus B| - 1$. As in Case C, we can apply Theorem 1.1 to $A' + B'$ to conclude

$$|P \setminus A| = |P_{A'} \setminus A'| \leq r \quad \text{and} \quad |P \setminus B| = |P_{B'} \setminus B'| + 1 \leq r + 1.$$

Moreover,

$$|A| + |P \setminus A| = |B| + |P \setminus B|.$$

Thus, if $|A| \geq |B|$, then we have $|P \setminus A| = |P \setminus B| - (|A| - |B|) \leq r + 1 - (|A| - |B|)$, while if $|B| \geq |A|$, then we instead find $|P \setminus B| = |P \setminus A| - (|B| + |A|) \leq r - (|B| - |A|)$. In either case, one of the estimates $|P \setminus A| \leq r$ or $|P \setminus B| \leq r + 1$ can be improved by at least $|A| - |B| - 1$, yielding the desired bound

$$|P \setminus A| + |P \setminus B| \leq 2r + 2 - |A| - |B| = 2r'.$$

It remains to consider the case when $\pi_1\left(\max \psi_2(B_1)\right) < \pi_1\left(\min \psi_2(B_0)\right)$ and $\pi_1\left(\max \psi_2(B_0)\right) < \pi_1\left(\min \psi_2(B_1)\right)$.

In this case, $B' = B_0'$, $B'' = B_0''$, $A' = A_0' \cup A_1'$ with $|A_1'| = 1$, $A'' = A_0'' \cup A_1'$ with $|A_0''| = 1$, and

$$|P_{B'} \setminus B'| + |P_{B''} \setminus B''| = |P \setminus B| + |P_B \setminus B| - 2. \quad (27)$$
As in Case C, we can apply Theorem 1.1 to \( A' + B' \) and \( A'' + B'' \) to conclude \( |P_{B'} \setminus B'| \leq r - (|B| - \min\{|A|, |B|\}) \) and \( |P_{B''} \setminus B''| \leq r - (|B| - \min\{|A|, |B|\}) \). Thus \( P = P_A \) and (27) give
\[
|P \setminus A| + |P \setminus B| \leq 2r + 2 + |P_A \setminus A| - |P_B \setminus B| - 2(|B| - \min\{|A|, |B|\}) \tag{28}
\]
\[
= 2r + 2 + |P_A| - |P_B| - |A| - |B|.
\]
As explained at the start of the case, we have \( |P \setminus A| + |P \setminus B| = 2|P_A \setminus A| + |A| - |B| \leq 2r + |A| - |B| \) (with the latter inequality in view of (23)). Thus, if \( |B| \geq |A| \), then \( |P \setminus A| + |P \setminus B| \leq 2r - \left| \left| A \right| - \left| B \right| \right| = 2r' - 2 \) follows, as desired. So instead assume \( |A| > |B| \). Observe that
\[
|P \setminus B| = |P| - |B| = |P_A \setminus A| + |A| - |B|.
\]
Using this substitution in (28) together with \( |A| > |B| \) and \( P_A = P \) yields \( |P_A \setminus A| + |P_B \setminus B| \leq 2r + 2 - (|A| - |B|) = 2r' \), which together with (28) gives the desired conclusions.

**Case A.** Let \( h_0 = |P_{A_0} \setminus A_0| \leq |P_A \setminus A| \), let \( h_1 = |P_{B_1} \setminus B_1| \leq |P_B \setminus B| \) and let \( h_0' = |P_{B_0} \setminus B_0| \). We may assume \( P_{A_1} \neq P_{B_1} \), and thus \( P_{A_1} \subsetneq P_{B_1} \), else Case B applies and the proof is complete. We may also assume \( P_{B_0} \neq P_{A_0} \), and thus \( P_{B_0} \subsetneq P_{A_0} \), else swapping the roles of A and B results in Case B applying, completing the proof as before. In particular,
\[
\delta(A_0, B_0) = 0 \quad \text{and} \quad \delta(B_1, A_1) = 0.
\tag{29}
\]
Moreover, if \( |A_0| = 1 \), then \( |P_{A_0}| = 1 \), whence the case hypothesis forces \( P_{B_0} = P_{A_0} \), contrary to what we just assumed. Likewise, if \( |B_1| = 1 \), then \( |P_{B_1}| = 1 \), whence the hypotheses of Case A force \( P_{A_1} = P_{B_1} \), contrary to what we just assumed. Therefore we may assume
\[
|A_0| \geq 2 \quad \text{and} \quad |B_1| \geq 2.
\tag{30}
\]
Since \( P_{B_0} \subsetneq P_{A_0} \) and \( P_{A_1} \subsetneq P_{B_1} \), we also have
\[
\text{diam}(A_0) = |P_{A_0}| - 1 > |P_{B_0}| - 1 = \text{diam}(B_0) \quad \text{and} \quad
\text{diam}(B_1) = |P_{B_1}| - 1 > |P_{A_1}| - 1 = \text{diam}(A_1).
\tag{31}
\]
We claim that if the following inequality holds, then the proof is complete:
\[
|A + B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + h_1.
\tag{\ast}
\]
Indeed, since \( |P| = |P_{A_0} \cup P_{B_1}| = |A_0| + h_0 + |B_1| + h_1 \) and \( |A + B| = |A| + |B| + \min\{|A|, |B|\} - 2 + r \) (by hypothesis), \((\ast)\) implies \( |P| \leq \min\{|A|, |B|\} + 1 + r \). Thus
\[
|P \setminus A| + |P \setminus B| = 2|P| - |A| - |B|
\leq 2\min\{|A|, |B|\} + 2 + 2r - |A| - |B|
= 2r + 2 - \left| \left| A \right| - \left| B \right| \right| = 2r',
\]
which would give the desired bound.
Recall (30) and (29). Then, in view of (31), we can apply Theorem A(ii) (with $d = 1$) to $A_0 + B_0$ unless $\gcd(A_0 - A_0 + B_0 - B_0) \geq 2$. If $\gcd(A_0 - A_0 + B_0 - B_0) \geq 3$, then $\gcd(A_0 - A_0) \geq 3$. If $\gcd(A_0 - A_0 + B_0 - B_0) = 2$ and $\gcd(A_0 - A_0) \leq 2$, then $\gcd(A_0 - A_0) = 2$ and $2 \mid \gcd(B_0 - B_0)$. If $|B_0| = 1$, then $|A_0 + B_0| \geq |A_0| = |A_0| + 2|B_0| - 2$. In all other cases, we can apply Theorem A(ii) (with $d = 1$) to $A_0 + B_0$, with the result that one of the following cases holds:

(a) $\gcd(A_0 - A_0) \geq 3$, or $\gcd(A_0 - A_0) = 2$ and $\gcd(A_1 - A_1) = 2d_0 > 0$,

(b) $|A_0 + B_0| \geq |A_0| + 2|B_0| - 2$,

(c) $|A_0 + B_0| \geq |A_0| + |B_0| + h_0 - 1$.

Likewise applying Theorem A(ii) (with $d = 1$) to $B_1 + A_1$ gives three possibilities:

(a') $\gcd(B_1 - B_1) \geq 3$, or $\gcd(B_1 - B_1) = 2$ and $\gcd(A_1 - A_1) = 2d_1 > 0$,

(b') $|B_1 + A_1| \geq |B_1| + 2|A_1| - 2$,

(c') $|B_1 + A_1| \geq |B_1| + |A_1| + h_1 - 1$.

Using symmetries, we see this gives us six subcases depending on which of the three cases holds for $A_0 + B_0$ and $A_1 + B_1$. However we first make the following observations.

**Claim 1.** We cannot have both $\gcd(B_0 - B_0) \neq 1$ and $\gcd(B_1 - B_1) \neq 1$, nor $\gcd(A_0 - A_0) \neq 1$ and $\gcd(A_1 - A_1) \neq 1$.

**Proof.** If $\gcd(B_0 - B_0) \neq 1$ and $\gcd(B_1 - B_1) \neq 1$, then $|P_B \setminus B| \geq |B| - 2$, contradicting (25) and (23). Likewise, $\gcd(A_0 - A_0) \neq 1$ and $\gcd(A_1 - A_1) \neq 1$ would imply $|P_A \setminus A| \geq |A| - 2$, also contradicting (25) and (23). □

**Claim 2.** $|(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |(B_0 + A_1) \cap I|$ for any interval $I \subseteq e_2 + Z e_1$ with $|I| \leq \operatorname{diam} B_1$.

**Proof.** For any $x \in A_0$ with $x + \min B_1 < \min I$, we have $x + \min B_1 \in (A_0 + B_1) \setminus I$. For any $x \in A_0$ with $x + \min B_1 \geq \min I$, we have

$$x + \max B_1 \geq \min I + \max B_1 - \min B_1 > \max I,$$

with the second inequality in view of the hypothesis $\max I - \min I + 1 = |I| \leq \operatorname{diam} B_1 = \max B_1 - \min B_1$. Thus $x + \max B_1 \in (A_0 + B_1) \setminus I$. As a result, there are at least $|A_0|$ elements contained in $(A_0 + B_1) \setminus I$, and the desired lower bound readily follows. □

**Subcase 1:** $\gcd(A_0 - A_0) \geq 2$ and $\gcd(B_1 - B_1) \geq 2$.

In this case, Claim 1 implies $\gcd(A_1 - A_1) = 1$ and $\gcd(B_0 - B_0) = 1$. Consequently, Corollary 3.1 gives $|B_1 + A_1| \geq 2|B_1| + |A_1| - 2$ and $|A_0 + B_0| \geq 2|A_0| + |B_0| - 2$. Combining these estimates with Theorem C applied to $A_1 + B_0$, we discover that $|A + B| \geq |A_0 + B_0| + |B_1 + A_1| + |A_1 + B_0| \geq 2|A| + 2|B| - 5$, contradicting (22).

**Subcase 2:** (b) holds for $A_0 + B_0$ and (b') holds for $A_1 + B_1$.

In this case, the estimates given by (b) and (b') combined with Theorem C applied to $A_0 + B_1$ imply $|A + B| \geq |A_0 + B_0| + |B_1 + A_1| + |A_0 + B_1| \geq 2|A| + 2|B| - 5$, contradicting (22).
Subcase 3: (c) holds for $A_0 + B_0$ and (c') holds for $A_1 + B_1$.

In this case, combining the estimates from (c) and (c') with Theorem C applied to $A_0 + B_1$ yields $|A + B| \geq |A_0 + B_0| + |B_1 + A_1| + |A_0 + B_1| \geq |A| + |B| - h_0 + |A_0| + h_1 + |B_1|$, establishing (★), thus completing the proof as already noted.

Subcase 4: (b) holds for $A_0 + B_0$ and gcd($B_1 - B_1$) ≥ 2, or (b') holds for $A_1 + B_1$ and gcd($A_0 - A_0$) ≥ 2.

By symmetry, we may w.l.o.g. assume the former case occurs. Thus

\[ |A_0 + B_0| \geq |A_0| + 2|B_0| - 2 \quad (32) \]

and gcd($B_1 - B_1$) ≥ 2. Hence Claim 1 implies gcd($B_0 - B_0$) = 1. In view of Subcase 1, we may assume gcd($A_0 - A_0$) = 1. Thus Corollary 3.1 gives

\[ |A_0 + B_1| \geq |A_0| + 2|B_1| - 2. \quad (33) \]

Likewise, if gcd($A_1 - A_1$) ≠ 1, then Corollary 3.1 gives $|B_0 + A_1| \geq |B_0| + 2|A_1| - 2$. On the other hand, if gcd($A_1 - A_1$) = 1, then Corollary 3.1 instead gives $|A_1 + B_1| \geq |A_1| + 2|B_1| - 2$. In consequence,

\[ |A_1 + B_1| \geq |A_1| + 2|B_1| - 2 \quad or \quad (34) \]
\[ |B_0 + A_1| \geq |B_0| + 2|A_1| - 2. \quad (35) \]

We will derive a contradiction in either case.

Suppose first that (34) holds. Then combining this estimate together with (32) and (33) yields

\[ |A + B| \geq |A_0 + B_0| + |A_1 + B_1| + |A_0 + B_1| \geq |A| + 2|B| - 4 + (|A_0| + 2|B_1| - 2). \]

On the other hand, combining (32) and (34) together with Theorem C applied to $B_0 + A_1$ yields

\[ |A + B| \geq |A_0 + B_0| + |A_1 + B_1| + |B_0 + A_1| \geq |A| + 2|B| - 4 + (|A| + |B| - |A_0| - |B_1| - 1). \]

Averaging these two estimates, we find that

\[ |A + B| \geq |A| + 2|B| - 4 + \frac{|A| + |B| - 3}{2} + \frac{|B_1|}{2} \geq \frac{3}{2}|A| + \frac{5}{2}|B| - 5, \]

where the final estimate follows from (30). But this contradicts (22). So instead assume (35) holds.

Combining (35) together with (32) and Theorem C applied to $A_1 + B_1$ yields

\[ |A + B| \geq 3|A| + 3|B| - 5 - 2|B_1| - 2|A_0|. \]

On the other hand, combining (32), (33) and Theorem C applied to $A_1 + B_1$ yields

\[ |A + B| \geq |A| + 2|B| - 5 + |A_0| + |B_1|. \]
Averaging 1 copy of the first bound with two copies of the second, we find that

$$|A + B| \geq \frac{5|A| + 7|B|}{3} - 5,$$

contrary to (22), which completes Subcase 4.

In view of the previous subcases, we can w.l.o.g. assume (c) holds for $A_0 + B_0$ but not (b) nor $\gcd(A_0 - A_0) \geq 2$. Thus

$$\gcd(A_0 - A_0) = 1 \quad \text{and} \quad |A_0 + B_0| \leq |A_0| + 2|B_0| - 3. \quad (36)$$

In consequence, in view of (31) and (29), we can apply Theorem A(ii) (with $d = 1$) to $A_0 + B_0$ and use the refined machinery described in Section 2. In particular, letting $J_{A_0} \subseteq P_{A_0}$ and $J_{B_0} \subseteq P_{B_0}$ be the intervals defined there, we have

1. $|A_0 + B_0| \geq |A_0| + |B_0| - 1 + \max\{h_0 + |J_{B_0} \setminus B_0|, \ h'_0 + |J_{A_0} \setminus A_0|\}$, \quad (37)
2. $|J_{B_0}| \geq |B_0| - h_0 + |J_{A_0} \setminus A_0| + |J_{B_0} \setminus B_0|$, \quad and \quad (38)
3. $|J_{A_0}| \geq |A_0| - h'_0 + |J_{A_0} \setminus A_0| + |J_{B_0} \setminus B_0|$. \quad (39)

We proceed with two claims before the next subcase. Let

$$R_1 \subseteq e_2 + Z e_1 \text{ be an interval with } |R_1| = \text{diam } B_1 \text{ that maximizes } |(B_0 + A_1) \cap R_1|.$$

**Claim 3.** We may assume $|J_{B_0}| < \text{diam } B_1 = |B_1| + h_1 - 1$ and $|(B_0 + A_1) \cap R_1| + |J_{B_0} \setminus B_0| < \text{diam } B_1 = |B_1| + h_1 - 1$, else $(\star)$ holds, completing the proof.

**Proof.** If $|J_{B_0}| \geq \text{diam } B_1$, then we can find an interval $J \subseteq J_{B_0}$ with $|J| = \text{diam } B_1$ and $|J_{B_0} \setminus B_0| \geq |J \setminus B_0|$, in which case $R = J + \min A_1 \subseteq e_2 + Z e_1$ is an interval with $|R| = \text{diam } B_1$ and $|(B_0 + A_1) \cap R| \geq |R| - |J_{B_0} \setminus B_0| = \text{diam } B_1 - |J_{B_0} \setminus B_0|$. On the other hand, if $|(B_0 + A_1) \cap R_1| + |J_{B_0} \setminus B_0| \geq \text{diam } B_1$, then the same conclusion holds for the interval $R = R_1 \subseteq e_2 + Z e_1$. In either case, applying Claim 2 yields

$$|(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + \text{diam } B_1 - |J_{B_0} \setminus B_0| = |A_0| + |B_1| + h_1 - 1 - |J_{B_0} \setminus B_0|.$$  

Combining this estimate together with (37) and the estimate $|A_1 + B_1| \geq |A_1| + |B_1| - 1$ obtained from applying Theorem C to $A_1 + B_1$, we find that

$$|A + B| \geq (|A_0| + |B_0| - 1 + h_0 + |J_{B_0} \setminus B_0|) + (|A_0| + |B_1| + h_1 - 1 - |J_{B_0} \setminus B_0|) + (|A_1| + |B_1| - 1) = |A| + |B| - 3 + |A_0| + |B_1| + h_0 + h_1,$$

yielding $(\star)$. \hfill \square

**Claim 4.** $|(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |(B_0 + A_1) \cap R_1| \geq |A_0| + |B_0| - h_0 + |J_{A_0} \setminus A_0|$.  

Proof. Applying Claim 2 with $I = R_1$ yields $|(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |(B_0 + A_1) \cap R_1|$. In view of Claim 3 and (38), we have $|(B_0 + A_1) \cap R_1| \geq |(B_0 + A_1) \cap (J_{B_0} + \min A_1)| \geq |J_{B_0} \cap B_0| - |J_{B_0} \setminus B_0| \geq |B_0| - h_0 + |J_{A_0} \setminus A_0|$, which combined with the previous inequality gives the desired bound.

Subcase 5: (b') holds for $A_1 + B_1$ with $\gcd(B_1 - B_1) \leq 2$.

Since (b') holds for $A_1 + B_1$, we have

$$|A_1 + B_1| \geq 2|A_1| + |B_1| - 2. \quad (40)$$

In view of (31), we have $|A_0| + h_0 > |B_0|$. Consequently, if $|A_0 + B_1| \geq 2|A_0| + |B_1| - 3$, then (37) and (40) imply

$$|A + B| \geq 2|A| + |B| - 6 + |A_0| + h_0 + |B_1| \geq 2|A| + 2|B| - 5,$$

contradicting (22). Therefore we may assume

$$|A_0 + B_1| \leq 2|A_0| + |B_1| - 4. \quad (41)$$

On the other hand, if $|A_0 + B_1| \geq |A_0| + |B_1| - 1 + h_1 - |J_{B_0} \setminus B_0|$, then (37) and Theorem C applied to $A_1 + B_1$ show that (★) holds, completing the proof. Thus, in view of (36), (41) and the subcase hypothesis $\gcd(B_1 - B_1) \leq 2$, applying Theorem A(ii) (with $d = 1$) to $A_0 + B_1$ implies that

$$\text{diam } A_0 > \text{diam } B_1 \quad \text{and} \quad |A_0 + B_1| \geq |A_0| + 2|B_1| - 2. \quad (42)$$

(Note diam $A_0 >$ diam $B_1$ implies $\delta(A_0, B_1) = 0$.)

Suppose $|A_0 + B_0| \geq 2|A_0| + |B_0| - 3$. Then combining this estimate with (40) and (42) yields

$$|A + B| \geq 2|A| + |B| - 7 + |A_0| + 2|B_1|.$$ 

Using the estimate $|A_0| + h_0 > |B_0|$ (from (31)) in (37) and combining the resulting bound with (40) and Theorem C applied to $A_1 + B_0$ yields

$$|A + B| \geq 3|A| + 3|B| - 3|A_0| - 2|B_1| - 3.$$ 

Averaging 1 copy of the above bound with 3 copies of the previous bound yields

$$|A + B| \geq \frac{9}{4} |A| + \frac{6}{4} |B| - 6 + |B_1| \geq \frac{11}{4} |A| + \frac{11}{4} |B| - 4,$$

where (30) was used for the final inequality, which contradicts (21). So we may instead assume $|A_0 + B_0| \leq 2|A_0| + |B_0| - 4$, which in view of (37) forces

$$h_0 \leq |A_0| - 3.$$ 

Let $I \subseteq P_{A_0}$ be an interval with $|I| = \text{diam } B_1$ such that $\gcd(A_0' + B_1 - A_0 - B_1) = 1$, where

$$A_0' = A_0 \cap I.$$
To see why such an interval $I$ exists, note that (42) implies $|P_{A_0}| = \text{diam } A_0 + 1 > \text{diam } B_1 + 1 > |I|$. Consequently, if $\gcd(B_1 - B_1) = 1$, then we need only choose $I$ so that $A_0'$ is nonempty (possible as $|I| = \text{diam } B_1 \geq 1$ by (30)), while if $\gcd(B_1 - B_1) = 2$, then $|I| = \text{diam } B_1 \geq 2$, so that the only way $I$ could fail to exist would be if there were no consecutive elements in $A_0$. However, that would contradict $h_0 \leq |A_0| - 3$, so $I$ exists. We have

$$|A_0'| \geq |I| - h_0 = |B_1| + h_1 - h_0 - 1.$$  \hspace{1cm} (43)

As argued in Claim 3, for any $x \in A_0$ with $x > \max I$, we have $x + \max B_1 > \max (I + P_{B_1})$, and for any $x \in A_0$ with $x < \min I$, we have $x + \min B_1 < \min (I + P_{B_1})$. Consequently, there are at least $|A_0| - |A_0'|$ elements of $A_0 + B_1$ that lie outside the interval $I + P_{B_1}$, thus being distinct from any element of $A_0' + B_1 \subseteq I + P_{B_1}$. Hence

$$|A_0 + B_1| \geq |A_0' + B_1| + |A_0| - |A_0'|.$$  \hspace{1cm} (44)

Since $|I| = \text{diam } B_1 < \text{diam } B_1 + 1 = |P_{B_1}|$, we have $\text{diam } B_1 > \text{diam } A_0'$, in turn implying $\delta(B_1, A_0') = 0$. Thus, since $\gcd(B_1 - B_1) \leq 2$ by subcase hypothesis, and since we also have $\gcd(A_0' + B_1 - A_0' - B_1) = 1$ as shown above, we can apply Theorem A(ii) (with $d = 1$) to $A_0' + B_1$ to conclude that either $|A_0' + B_1| \geq 2|A_0'| + |B_1| - 2$ or $|A_0' + B_1| \geq |A_0'| + |B_1| - 1 + h_1$. If the latter holds, then (44) implies $|A_0 + B_1| \geq |A_0| + |B_1| - 1 + h_1$. Combined with (37) and the bound $|A_1 + B_1| \geq |A_1| + |B_1| - 1$ (from Theorem C), we obtain the estimate $|A + B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + h_1$, establishing (★) and completing the proof. So we may instead assume $|A_0' + B_1| \geq 2|A_0'| + |B_1| - 2$, which combined with (44) and (43) gives

$$|A_0 + B_1| \geq |A_0| + 2|B_1| - 3 + h_1 - h_0.$$  \hspace{1cm} (45)

Combining the estimates in (37), (45) and (40) gives

$$|A + B| \geq 2|A| + |B| - 6 + 2|B_1| + h_1.$$  \hspace{1cm} (46)

Combining the estimates in (37), (42) and (40) gives

$$|A + B| \geq 2|A| + |B| - 5 + 2|B_1| + h_0.$$  \hspace{1cm} (47)

Combining the estimates in (37), Claim 4 and (40) gives

$$|A + B| \geq 2|A| + 2|B| - 3 + |B_1|.$$  \hspace{1cm} (48)

From (47), we deduce that

$$h_1 \geq |B_1| - 1 + |A_1| \geq |B_1|,$$

else (★) holds, as desired, and applying this estimate in (46) gives

$$|A + B| \geq 2|A| + |B| - 6 + 3|B_1|.$$  \hspace{1cm} (49)

Averaging 3 copies of the bound in (48) with 1 copy of the bound in (49), we obtain

$$|A + B| \geq 2|A| + \frac{7}{4}|B| - \frac{15}{4} \geq |\tilde{A}| + \frac{11}{4}|\tilde{B}| - \frac{15}{4}.$$
contrary to (21), which completes Subcase 5.

**Subcase 6:** $\gcd(B_1 - B_1) = d \geq 2$. Moreover, $\gcd(A_1 - A_1) = 2d_1 > 0$ when $d = 2$.

(Note: the case $\gcd(B_1 - B_1) = 2$ with $|A_1| = 1$ was covered in Subcase 5 as (b') holds trivially when $|A_1| = 1$.)

In this case, $B_1$ is contained entirely in the $d\mathbb{Z}e_1$-coset $\min B_1 + d\mathbb{Z}e_1$. Let

$$h^s_i = |P_{B_1} \cap (\min B_1 + d\mathbb{Z}e_1)| - |B_1|$$

and observe that

$$h_1 = (|B_1| + h^s_1 - 1)(d - 1) + h^s_1 = dh^s_1 + (d - 1)(|B_1| - 1).$$  \hspace{1cm} (50)

Since $|A_0| + h_0 > |P_{B_0}| \geq |B| - |B_1|$ (by (31)), we have

$$h_0 \geq |B| - |A_0| - |B_1| + 1. \hspace{1cm} (51)$$

In view of (36), let $s \in [2, d]$ be the number of $d\mathbb{Z}e_1$-cosets that intersect $A_0$, let $x_1, \ldots, x_s \in A_0$ be representatives for these cosets, and partition $A_0 = \bigcup_{i=1}^{s} A^i_0$ with each $A^i_0 = (x_i + d\mathbb{Z}e_1) \cap A_0$. We may w.l.o.g. assume $x_1 = \min A_0$ and that we have ordered the $x_i$ in increasing cyclic order modulo $de_1$.

The quantity $h_0 = |P_{A_0} \setminus A_0|$ counts the number of holes in $A_0$, i.e., the number of elements $x \in P_{A_0} \setminus A_0$. We can more precisely count these holes as follows. Let $h^s_0$ be the number of $x \in P_{A_0} \setminus A_0$ with $x \in x_i + d\mathbb{Z}e_1$. We have max $A_0 \equiv x_{s-\epsilon} \pmod{de_1}$ for some $\epsilon \in [0, s-1]$. Let $h^s_0 = h^s_0 + \ldots + h^s_0$. Let $\rho$ be the number of $x \in P_{A_0} \setminus A_0$ not counted by any $h^s_0$ (so $x \not\in x_i + d\mathbb{Z}e_1$ for all $i$) that also lie between max $A_0$ and the largest element of $P_{A_0}$ from min $A_0 + d\mathbb{Z}e_1$. Note $\rho \in [0, d-s]$ with $\rho = 0$ occurring precisely when the elements $x_1, \ldots, x_{s-\epsilon}$ form an arithmetic progression with difference $e_1$ modulo $de_1$. With this notation, we now have

$$h_0 = h^s_0 + \frac{|A_0| + h^s_0 + \epsilon - s}{s}(d - s) + \rho \geq \frac{|A_0| - s}{s}(d - s) + \frac{d}{s}h^s_0. \hspace{1cm} (52)$$

generalizing the formula from (50). When $d = s$, the estimate in (52) is trivial without some bound for $h^s_0$, meaning we will need a better way to deal with estimating $h^s_0$ when $s = d$. One way will be through the following setup.

For $i \in [1, s]$, we can apply Theorem A(i) (with $d$ as defined above) and Theorem C to $A^i_0 + B_1$ to conclude

$$|A^i_0 + B_1| \geq |A^i_0| + |B_1| - 1 + \Omega_i, \hspace{0.5cm} \text{where } \Omega_i = \max\{0, \min\{h^s_i, |B_1| - 2, |A^i_0| - 2\}\}.$$

Thus we obtain

$$|A_0 + B_1| = \sum_{i=1}^{s}|A^i_0 + B_1| \geq |A_0| + s(|B_1| - 1) + \sum_{i=1}^{s}\Omega_i \geq |A_0| + s(|B_1| - 1). \hspace{1cm} (53)$$

Let $\Theta \subseteq [1, s]$ be a subset of indices $\alpha \in [1, s]$ with $|A^0_\alpha| - 2 < \min\{h^s_\alpha, |B_1| - 2\}$ for $\alpha \in \Theta$. Note this ensures $\Omega_\alpha \geq |A^0_\alpha| - 2$ for $\alpha \in \Theta$. 


Each $P_{A_0} \cap (x_i + d \mathbb{Z} e_1)$ for $i \in [1, s - \epsilon]$ has size $|A_0| + h_0^\epsilon + \epsilon$, while each $P_{A_0} \cap (x_i + d \mathbb{Z} e_1)$ for $i \in [s - \epsilon + 1, s]$ has size $|A_0| + h_0^\epsilon + \epsilon - 1$. Thus $h_0^\epsilon = \frac{|A_0| + h_0^\epsilon + \epsilon}{s} - |A_0|$ for $i \in [1, s - \epsilon]$, and $h_0^1 = \frac{|A_0| + h_0^\epsilon + \epsilon}{s} - |A_0| - 1$ for the $\epsilon$ indices $i \in [s - \epsilon + 1, s]$. Moreover, in view of Subcase 5, we can assume (b') does not hold when $|\alpha| \leq 1$. Claim 1 implies $\gcd(\sum_{\alpha \in \Theta} |A_0^\alpha| - \gcd(\sum_{\alpha \in \Theta} |A_0^\alpha| - \sum_{\alpha \in \Theta} |A_0^\alpha| = \frac{|\Theta|}{s - |\Theta|} |A_0| - \epsilon h_0^\epsilon + \epsilon).$ In consequence, assuming $|\Theta| < s$, we have

$$h_0^\epsilon + |\Theta| \geq \sum_{\alpha \in \Theta} h_0^\alpha \min\{|\epsilon|, |\Theta|\} = \frac{|A_0| - \sum_{\alpha \in \Theta} |A_0^\alpha|}{s - |\Theta|} |\Theta| - \sum_{\alpha \in \Theta} |A_0^\alpha| = \frac{|\Theta|}{s - |\Theta|} |A_0| - \frac{s}{s - |\Theta|} \sum_{\alpha \in \Theta} |A_0^\alpha|. \quad (54)$$

Since $\sum_{\alpha \in \Theta} \alpha \geq \sum_{\alpha \in \Theta} (|A_0^\alpha| - 2) and |A_0^\alpha| - 2 < \min\{h_0^\epsilon, |B_1| - 2\} for \alpha \in \Theta$, we can combine these estimates with (54) to obtain

$$h_0^\epsilon + \sum_{\alpha \in \Theta} \alpha \geq \frac{|\Theta|}{s - |\Theta|} (|A_0| - \sum_{\alpha \in \Theta} |A_0^\alpha|) - 3|\Theta| \geq \frac{|\Theta|}{s - |\Theta|} (|A_0| - |\Theta|(|B_1| - 1)) - 3|\Theta|, \quad (55)$$

$$h_0^\epsilon + \sum_{\alpha \in \Theta} \alpha \geq \frac{|\Theta|}{s - |\Theta|} (|A_0| - \sum_{\alpha \in \Theta} |A_0^\alpha|) - 3|\Theta| \geq \frac{|\Theta|}{s - |\Theta|} (|A_0| - |\Theta|(|h_0^\epsilon + 1)) - 3|\Theta|. \quad (56)$$

If $d = 2$, then the subcase hypotheses ensures $\gcd(B_1 - B_1) = 2$ and $\gcd(A_1 - A_1) = 2d_1 > 0$, whence Claim 1 implies $\gcd(A_0 - A_0) = \gcd(B_0 - B_0) = 1$. Thus, Corollary 3.1 applied to $B_0 + A_1$ gives

$$|B_0 + A_1| \geq |B_0| + 2|A_1| - 2 \quad \text{when } d = 2. \quad (57)$$

Moreover, in view of Subcase 5, we can assume (b') does not hold when $d = 2$, in which case $|B_1 + A_1| \leq |B_1| + 2|A_1| - 3$. Consequently, we can apply Theorem A(ii) (with $d = 2$) to $B_1 + A_1$ to conclude (in view of (29) and (31)) that

$$|B_1| + 2|A_1| - 3 \geq |A_1 + B_1| \geq |A_1| + |B_1| - 1 + h_0^\epsilon \quad \text{when } d = 2. \quad (58)$$

If $\Omega_0 \geq h_0^\epsilon$ for some $\alpha \in [1, 2]$, then combining (37), (53) and (58) yields $|A + B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + (|B_1| - 1 + 2h_0^\epsilon).$ Hence ($\star$) holds in view of (50), as desired. So we may assume

$$\Omega_0 < h_0^\epsilon \quad \text{for all } \alpha \in [1, 2] \text{ when } d = 2. \quad (59)$$

We continue with the following claim.

**Claim 5.** $|A_1 + B_1| \leq |A_1| + 2|B_1| - 3$, else ($\star$) holds, as desired.

**Proof.** Assume to the contrary that

$$|A_1 + B_1| \geq |A_1| + 2|B_1| - 2. \quad (60)$$

From (37), (60) and $|B_0 + A_1| \geq |B_0| + |A_1| - 1$ (by Theorem C), we have

$$|A + B| \geq 2|A| + 2|B| - 4 - |A_0| + h_0. \quad (61)$$
From (37), Claim 4 and (60), we have
\[ |A + B| \geq |A| + 2|B| - 3 + |A_0|. \] (62)

Suppose \( d = 2 \), so \( s = d = 2 \). Then combining (37), (51), (57), and (60) yields
\[ |A + B| \geq 3|A| + 3|B| - 4 - 3|A_0| - |B_1|. \] (63)

If \( \Omega_\alpha \geq |A_0^\alpha| - 2 \) for all \( \alpha \in [1, 2] \), then \( \sum_{i=1}^2 \Omega_i \geq |A_0| - 4 \), whence (53) implies \( |A_0 + B_1| \geq 2|A_0| + 2|B_1| - 6 \). Combining this estimate with (37), (51) and (60) yields \( |A + B| \geq |A| + 2|B| - 8 + |A_0| + 2|B_1| \). Averaging 1 copy of this bound with 2 copies of (63) and 5 copies of (62) yields
\[ |A + B| \geq \frac{12|A| + 18|B| - 31}{8} \geq |\tilde{A}| + \frac{11}{4}|\tilde{B}| - \frac{31}{8}, \]
contrary to (21). Therefore, in view of (59), we may assume \( \Omega_{\gamma'} \geq |B_1| - 2 \) for some \( \gamma' \in [1, 2] \).

Let \( \gamma \) be the other index from \([1, 2]\). If \( \Omega_{\gamma} < |B_1| - 2 \), then in view of (59), we have \( |A_0^\gamma| - 2 \leq \Omega_{\gamma} \leq \min\{h_1^\gamma, |B_1| - 2\} \), allowing us to use (54)–(56) with \( \Theta = \{\gamma\} \). In such case, the estimates (37), (53), \( \Omega_{\gamma'} \geq |B_1| - 2 \), \( h_0 \geq h_0^\gamma \) and (55) give
\[ |A_0 + B_0| + |A_0 + B_1| \geq (|A_0| + |B_0| - 1 + h_0) + (|A_0| + 2|B_1| - 2 + \Omega_\gamma + \Omega_{\gamma'}) \]
\[ \geq |B| + 2|A_0| + |B_1| - 3 + h_0^\gamma + \Omega_\gamma + (|B_1| - 2) \]
\[ \geq |B| + 3|A_0| + |B_1| - 7. \]

Combining this estimate with (60) yields \( |A + B| \geq |A| + |B| - 9 + 2|A_0| + 3|B_1| \). Averaging 1 copy of this bound with 3 copies of (63) and 7 copies of (62) yields
\[ |A + B| \geq \frac{17|A| + 24|B| - 42}{11} \geq |\tilde{A}| + \frac{30}{11}|\tilde{B}| - \frac{42}{11}, \]
contrary to (21). Therefore we instead conclude that \( \Omega_\gamma \geq |B_1| - 2 \). Since we already showed \( \Omega_{\gamma'} \geq |B_1| - 2 \), (53) implies that \( |A_0 + B_1| \geq |A_0| + 4|B_1| - 6 \). Combining this estimate with (37) and (58) yields
\[ |A + B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + (3|B_1| - 5 + h_1^\gamma). \]

Thus we may assume \( 3|B_1| - 5 + h_1^\gamma \leq h_1 - 1 = 2h_1^\gamma + |B_1| - 2 \), with the latter inequality in view of (50), else (★) holds, as desired. Hence \( h_1^\gamma \geq 2|B_1| - 3 \). Combining (37), (51), (57), (58) and \( h_1^\gamma \geq 2|B_1| - 3 \) yields
\[ |A + B| \geq 3|A| + 3|B| - 6 - 3|A_0|. \] (64)

From (37), Claim 4, (58), \( h_1^\gamma \geq 2|B_1| - 3 \) and (30), we have
\[ |A + B| \geq |A| + 2|B| - 5 + |A_0| + |B_1| \geq |A| + 2|B| - 3 + |A_0|. \] (65)

Averaging 3 copies of (65) with 1 copy of (64) yields
\[ |A + B| \geq \frac{6|A| + 9|B| - 15}{4} \geq |\tilde{A}| + \frac{11}{4}|\tilde{B}| - \frac{15}{4}. \]
contrary to (21), which completes the case when \( d = 2 \). So we may now assume \( d \geq 3 \).

We next show that

\[
|(A_0 + B_1) \cup (B_0 + A_1)| < |A_0| + |B_0|. 
\]  
(66)

Suppose (66) fails. Then combining the resulting bound with (37) and (60) yields

\[
|A + B| \geq |A| + 2|B| - 3 + |A_0| + h_0.
\]  
(67)

If \( |A_0 + B_1| \geq |A_0| + 3|B_1| - 3 \), then (37), (60) and (51) give \( |A + B| \geq |A| + 2|B| - 5 + 3|B_1| \). Averaging this estimate with 3 copies of the bound from (67) (using (51) to estimate \( h_0 \)) yields

\[
|A + B| \geq |A| + \frac{11}{4}|B| - \frac{11}{4},
\]

contrary to (21). Therefore, instead assume \( |A_0 + B_1| \leq |A_0| + 3|B_1| - 4 \), whence (53) implies that \( s = 2 \) with \( \Omega_\alpha < |B_1| - 2 \) for all \( \alpha \). Thus (52) implies \( h_0 \geq \frac{1}{2}|A_0| - 1 \) in view of \( d \geq 3 \). But then, using this estimate in (67) and (61) and averaging the resulting bound from (67) with 3 copies of the resulting bound from (61) yields

\[
|A + B| \geq \frac{7|A| + 8|B| - 19}{4} \geq |A| + \frac{11}{4}|B| - \frac{19}{4},
\]

contrary to (21). This shows that (66) holds.

Next, we show that

\[
\text{diam } B_0 \geq \text{diam } B_1 > \text{diam } A_1.
\]  
(68)

The latter inequality follows from (31). As a result, if (68) fails, then \( |P_{B_0}| - 1 = \text{diam } B_0 < \text{diam } B_1 \), whence we can apply Claim 2 with \( I = P_{B_0} + \min A_1 \) to obtain \( |(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |B_0| \), contradicting (66). This establishes (68), in turn implying \( \delta(B_0, A_1) = 0 \).

In view of Claim 1 and \( \gcd(B_1 - B_1) = d \geq 3 \), we have \( \gcd(B_0 - B_0) = 1 \). We claim that

\[
|B_0 + A_1| \geq |B_0| + 2|A_1| - 2 = 2|A| + |B| - 2 - 2|A_0| - |B_1|. 
\]  
(69)

Indeed, if this fails, then (68) and \( \gcd(B_0 - B_0) = 1 \) allow us to apply A(ii) (with \( d = 1 \)) to \( B_0 + A_1 \) to conclude that there is an interval \( R \subseteq B_0 + A_1 \subseteq P_{B_0} + P_{A_1} \subseteq P_{A_0} + P_{B_1} \) with \( |R| \geq \text{diam } B_1 \), then we can find an interval \( I \subseteq R \subseteq B_0 + A_1 \) with \( |I| = \text{diam } B_1 \). Applying Claim 2 then yields \( |(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |I| = |A_0| + |B_1| + h_1 - 1 \), which combined with (37) and \( |A_1 + B_1| \geq |A_1| + |B_1| - 1 \) (from Theorem C) yields (\( \star \)), as desired. On the other hand, if \( |R| < \text{diam } B_1 \), then we can apply Claim 2 with \( I = R \) to find \( |(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |R| \geq |A| + |B_0| - 1 \). Combining this estimate with (37) and (60) then yields \( |A + B| \geq 2|A| + 2|B| - 4 + h_0, \) contrary to (22). This establishes (69).

From (37), (51), (53) and (60), we have

\[
|A + B| \geq |A| + 2|B| - 2 + s(|B_1| - 1) + \sum_{i=1}^{s} \Omega_i \geq |A| + 2|B| - 4 + 2|B_1|. 
\]  
(70)
From (37), (51), (69) and (60), we have

\[ |A + B| \geq 3|A| + 3|B| - 4 - 3|A_0| - |B_1|. \]  

(71)

We claim that

\[ |A + B| < |A| + 2|B| - 13 + 5|B_1|. \]  

(72)

Indeed, if (72) fails, then averaging 1 copy of the resulting bound with 15 copies of the bound in (62) and 5 copies of the bound in (71) yields

\[ |A + B| \geq \frac{31|A| + 47|B| - 78}{21} \geq |\tilde{A}| + \frac{19}{7}|\tilde{B}| - \frac{26}{7}, \]

contradicting (21). This establishes (72). But now (72) and (70) ensure that

\[ s \leq 4. \]

Suppose \( s \leq d - 1 \). Then (52) implies that \( h_0 \geq \frac{|A_0|}{s} - 1 \). Combining this estimate with (37), (69) and (60) gives

\[ |A + B| \geq 3|A| + 2|B| - 6 - \frac{2s - 1}{s}|A_0|. \]  

(73)

Averaging \( s \) copies of the bound in (73) with \( 2s - 1 \) copies of the bound in (62) yields

\[ |A + B| \geq \frac{(5s - 1)|A| + (6s - 2)|B| - 12s + 3}{3s - 1} \geq |\tilde{A}| + \frac{8s - 2}{3s - 1}|\tilde{B}| - \frac{12s - 3}{3s - 1}. \]

However, for the values \( s = 2, 3, 4 \), the above bound becomes \( |A + B| \geq |\tilde{A}| + \frac{14}{5}|\tilde{B}| - \frac{21}{5}, \)

\( |A + B| \geq |\tilde{A}| + \frac{32}{7}|\tilde{B}| - \frac{33}{7} \) and \( |A + B| \geq |\tilde{A}| + \frac{30}{11}|\tilde{B}| - \frac{45}{11} \), respectively, all of which contradict (21). So we may instead assume \( s = d \geq 3 \). Thus

\[ s = d = 3 \quad \text{or} \quad s = d = 4. \]

From (37), (53), (60), (50) and \( s = d \), we find that

\[ |A + B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + s(|B_1| - 1) + \sum_{i=1}^{s} \Omega_i \]  

(74)

\[ = |A| + |B| - 3 + |A_0| + |B_1| + h_0 + h_1 + (|B_1| - 1) + \sum_{i=1}^{s} \Omega_i - s h_1^s. \]

Consequently, we can assume

\[ \sum_{i=1}^{s} \Omega_i \leq s h_1^s - |B_1|, \]  

(75)

else (\( \star \)) holds, as desired. In particular, \( h_1^s \geq \frac{1}{2}|B_1| > 0 \). Also, (70) and (72) imply that \( |B_1| \geq \frac{12 - s}{5 - s} \geq s + 1 \geq 4 \) for \( s \in [3, 4] \), whence

\[ h_1^s \geq 2 \quad \text{and} \quad |B_1| - 2 \geq 2. \]  

(76)

Let us next show that there must be some \( \gamma \in [1, s] \) with \( \Omega_\gamma < |A_0^\gamma| - 2 \). If this fails, then \( \sum_{i=1}^{s} \Omega_i \geq |A_0| - 2s \). Using this estimate in (53) gives \( |A_0 + B_1| \geq 2|A_0| + s|B_1| - 3s \geq
For averaging 3 copies of the bound in (78) with 3 indices $i$ that contradict (21). So we may assume $\Omega_\gamma < |A_0^\gamma| - 2$ for some $\gamma \in [1, s]$, in which case (76) ensures that

$$\Omega_\gamma \geq \min\{h_1^\gamma, |B_1| - 2\} \geq 2.$$  

If $s = 4$, then (70) and (72) ensure that $\Omega_i < |B_1| - 2$ for all $i \in [1, s]$. If $\Omega_i \geq h_1^\gamma$ for at least two indices $i \in [1, s]$, then (75) implies that $2h_1^\gamma \leq \sum_{i=1}^s \Omega_i \leq 4h_1^\gamma - |B_1|$ in turn implying

$$|B_1| \leq 2h_1^\gamma \leq \sum_{i=1}^s \Omega_i,$$

whence (70) contradicts (72). So we see that there must be at least $s - 1$ indices $i \in [1, s]$ with $|A_0^\gamma| = 2 \leq \Omega_i < \min\{h_1^\gamma, |B_1| - 2\}$, allowing us to use (55) with $\Theta = [1, s] \setminus \{\gamma\}$ when $s = 4$. Let us next show that we can assume the same when $s = 3$. If this fails for $s = 3$, then either $\sum_{i=1}^s \Omega_i \leq 2|B_1| - 4$ or $\sum_{i=1}^s \Omega_i \geq |B_1| - 2 + h_1^\gamma$ or $\sum_{i=1}^s \Omega_i \geq 2h_1^\gamma$.

In the first case, (70) contradicts (72). In the second case, (75) implies $|B_1| - 2 + h_1^\gamma \leq \sum_{i=1}^s \Omega_i \leq 3h_1^\gamma - |B_1|$, in turn implying $h_1^\gamma \geq |B_1| - 1$, whence $\sum_{i=1}^s \Omega_i \geq |B_1| - 2 + h_1^\gamma \geq 2|B_1| - 3$, so that (70) again contradicts (72). Finally, in the third case, (75) instead implies $2h_1^\gamma \leq \sum_{i=1}^s \Omega_i \leq 3h_1^\gamma - |B_1|$, in turn implying $|B_1| \leq h_1^\gamma$, whence $\sum_{i=1}^s \Omega_i \geq 2h_1^\gamma \geq 2|B_1|$, so that (70) once more contradicts (72).

Thus, for both remaining values of $s$, we see that we can use (55) with $|\Theta| = s - 1$.

Now (74), (77) with $h_0 \geq h_0^\gamma$ and (55) with $|\Theta| = s - 1$ combine to yield

$$|A + B| \geq |A| + |B| + (s^2 - 6s + 3) + s|A_0| - (s^2 - 3s)|B_1|.$$  

Averaging 3 copies of the bound in (78) with $3s - 8 \geq 1$ (in view of $s \geq 3$) copies of the first bound in (70) (using the estimate $\sum_{i=1}^s \Omega_i \geq 0$) and $s$ copies of the bound in (71) yields

$$|A + B| \geq \frac{(6s - 5)|A| + (9s - 13)|B| - (20s - 25)}{4s - 5} \geq \left|\tilde{A}\right| + \frac{11s - 13}{4s - 5} \left|\tilde{B}\right| - \frac{20s - 25}{4s - 5},$$

For $s = 3, 4$, the respective bound given above is $|A + B| \geq |\tilde{A}| + \frac{20}{7}|\tilde{B}| - 5$ and $|A + B| \geq |\tilde{A}| + \frac{31}{11}|\tilde{B}| - 5$, both of which contradict (21), which at last completes the proof of Claim 5. □
Since \( \gcd(B_1 - B_1) = d \geq 2 \), Claim 1 implies that \( \gcd(B_0 - B_0) = 1 \). In view of Claim 5 and Corollary 3.1, it follows that \( A_1 \) is contained in a single \( d\mathbb{Z}_{e_1} \)-coset, i.e., \( d \mid \gcd(A_1 - A_1) \). In view of \( \gcd(B_0 - B_0) = 1 \), let \( t \in [2, d] \) be the number of \( d\mathbb{Z}_{e_1} \)-cosets that intersect \( B_0 \). Repeating the setup from the beginning of Subcase 6 for \( B_0 \) instead of \( A_0 \), we find that

\[
h^t_0 \geq \frac{|B_0| - t}{t}(d - t). \tag{79}
\]

Similar arguments can be used to estimate the quantity \( |J_{B_0} \setminus B_0| \), giving

\[
|J_{B_0} \setminus B_0| \geq \frac{|J_{B_0}| - |J_{B_0} \setminus B_0| - t}{t}(d - t) \geq \frac{|B_0| - h_0 + |J_{A_0} \setminus A_0| - t}{t}(d - t) \geq \frac{|B_0| - h_0 - t}{t}(d - t), \tag{80}
\]

where the second inequality follows from (38). Using (39) instead of (38), we can likewise estimate \( |J_{A_0} \setminus A_0| \), giving us

\[
|J_{A_0} \setminus A_0| \geq \frac{|J_{A_0}| - |J_{A_0} \setminus A_0| - s}{s}(d - s) \geq \frac{|A_0| - h'_0 + |J_{B_0} \setminus B_0| - s}{s}(d - s) \geq \frac{|A_0| - h'_0 - s}{s}(d - s). \tag{81}
\]

Applying Corollary 3.1 to \( B_0 + A_1 \), we obtain

\[
|B_0 + A_1| \geq |B_0| + t(|A_1| - 1). \tag{82}
\]

**Claim 6.** \( |A_1| + |B_1| + h_0^t - 1 \leq |A_1 + B_1| \leq |B_1| + 2|A_1| - 3 \).

**Proof.** When \( d = 2 \), the claim follows from (58). So we may assume \( d \geq 3 \). Since \( A_1 \) is contained in a \( d\mathbb{Z}_{e_1} \)-coset with \( \gcd(B_1 - B_1) = d \) and \( \text{diam}(B_1) > \text{diam}(A_1) \) (in view of (31)), it suffices in view of (29) to show \( |A_1 + B_1| \leq |B_1| + 2|A_1| - 3 \) as the remaining conclusion of the claim will then follow from applying Theorem A(ii) (with \( d = \gcd(B_1 - B_1) \)) to \( B_1 + A_1 \). Assuming by contradiction that

\[
|A_1 + B_1| \geq |B_1| + 2|A_1| - 2, \tag{83}
\]

we can combine (83), (37), and Claim 4 to yield

\[
|A + B| \geq 2|A| + |B| - 3 - |B_1|. \tag{84}
\]

Combining (83), (37), and (53) (using the estimates \( h_0 \geq 0 \)) yields

\[
|A + B| \geq 2|A| + |B| - (s + 3) + s|B_1|. \tag{85}
\]

Averaging \( s \) copies of the bound in (84) with 1 copy of the bound from (85) gives us

\[
|A + B| \geq \frac{(2s + 2)|A| + (2s + 1)|B| - (4s + 3)}{s + 1} \geq \frac{3s + 2}{s + 1} |B| - \frac{4s + 3}{s + 1}. \]

For \( s \geq 3 \), the above bounds yields \( |A + B| \geq |\tilde{A}| + \frac{11}{4}|\tilde{B}| - \frac{15}{4} \), contrary to (21). It remains to handle the case \( s = 2 \).
In this case, (52) implies \( h_0 \geq \frac{1}{2}|A_0| - 1 \) in view of \( d \geq 3 \). Using this estimate together with (83), (37), and (53) (and recalling that \( s = 2 \)) yields

\[
|A + B| \geq 2|A| - 6 + \frac{1}{2}|A_0| + 2|B_1|.
\]

From (37), (51), (83) and (82) (using \( t \geq 2 \)), we obtain

\[
|A + B| \geq 4|A| - 3|B| - 4 - 4|A_0| - 2|B_1|.
\]

(Averaging 14 copies of (84) with 8 copies of (86) and 1 copy of (87) yields

\[
|A + B| \geq \frac{48|A| + 39|B| - 94}{23} \geq |\tilde{A}| + \frac{64}{23}|\tilde{B}| - \frac{94}{23},
\]

contrary to (21). This completes Claim 6. \( \square \)

From (37), Claim 6 and Claim 4, we obtain

\[
|A + B| \geq |A| + |B| - 2 + |A_0| + |B_0| + \left(|J_{B_0} \setminus B_0| + |J_{A_0} \setminus A_0| + h_{1}^e\right).
\]

From (37), Claim 6 and the first estimate in (53), we obtain

\[
|A + B| \geq |A| + (s + 1)|B| - (s + 2) + |A_0| - s|B_0| + \left(h_0 + |J_{B_0} \setminus B_0| + h_{1}^e + \sum_{i=1}^{s} \Omega_i\right),
\]

\[
|A + B| \geq |A| + (s + 1)|B| - (s + 2) + |A_0| - s|B_0| + \left(h_0 + |J_{A_0} \setminus A_0| + h_{1}^e + \sum_{i=1}^{s} \Omega_i\right).
\]

From (37), Claim 6 and (82), we obtain

\[
|A + B| \geq (t + 1)|A| + |B| - (t + 2) - t|A_0| + |B_0| + \left(h_0 + |J_{B_0} \setminus B_0| + h_{1}^e\right),
\]

\[
|A + B| \geq (t + 1)|A| + |B| - (t + 2) - t|A_0| + |B_0| + \left(h_0 + |J_{A_0} \setminus A_0| + h_{1}^e\right).
\]

Let \( r = \min\{s, t\} \in [2, d] \). Since (89)–(92) are clearly monotonically increasing with \( s \) and \( t \), these bounds all hold replacing \( s \) or \( t \) by \( r \).

Now suppose that the following inequalities hold, where \( x \in [0, 1] \) and \( \epsilon \geq 0 \) are as yet unspecified real numbers:

\[
|A + B| \geq |A| + |B| - 2 + |A_0| + |B_0|, \\
|A + B| \geq |A| + (r + 1)|B| - (r + 2) + |A_0| - r|B_0| + x \min\{|A_0|, |B_0|\} - \epsilon, \\
|A + B| \geq (r + 1)|A| + |B| - (r + 2) - r|A_0| + |B_0| + x \min\{|A_0|, |B_0|\} - \epsilon.
\]

Observing that the system of inequalities given by (93)–(95) is completely symmetric with respect to the variables \( |A_0| \) and \( |B_0| \), it does not matter whether \( |A_0| \) or \( |B_0| \) attains the minimum in \( \min\{|A_0|, |B_0|\} \), and we will w.l.o.g. assume \( |A_0| = \min\{|A_0|, |B_0|\} \) for the following calculation (the case \( |B_0| = \min\{|A_0|, |B_0|\} \) being nearly identical): averaging \( r^2 - 1 - x(r + 1) \)

\[\]
copies of the bound in (93) with $r+1-x$ copies of the bound in (94) and $r+1+x$ copies of the bound in (95), we obtain

$$|A + B| \geq \left(\frac{(2r+1)(r+1-x)|A| + (2r+1)(r+1-x)|B| - 2(r+1)(2r+1-x + \epsilon)}{(r+1)(r+1-x)}\right)$$

$$\geq \frac{|\tilde{A}| + 3r + 1 - x}{r+1-x} |\tilde{B}| - \frac{2(2r+1-x + \epsilon)}{r+1-x}.$$  \hspace{1cm} (96)

Moreover, if $x < 1$ and any of the bounds in (93)--(95) is strict, then the bound in (96) will also be strict. The derivative with respect to $r$ of the above expression is $\frac{2(\epsilon + (1-x)(|\tilde{B}| - 1))}{(r+1-x)^2}$, which is non-negative for $x \in [0, 1]$, meaning the bound in (96) is minimized for small $r$. In view of (88), (89) and (91), the bounds (93)--(95) hold with $x = \epsilon = 0$. Consequently, if $r \geq 6$, then (96) with $x = \epsilon = 0$ gives

$$|A + B| \geq |\tilde{A}| + \frac{19}{r} |\tilde{B}| - \frac{26}{r},$$

contrary to (21), meaning we may now assume $r \leq 5$.

Suppose $d \geq r + 2$. Then (79) and (52) imply $h_0' \geq \frac{2}{r+1} |B_0| - 2 \geq (1 - \frac{r}{6}) |B_0| - \frac{3}{2}$ or $h_0 \geq \frac{2}{r} |A_0| - 2 \geq (1 - \frac{r}{6}) |A_0| - \frac{3}{2}$ (depending on whether $r = t$ or $r = s$), with the latter inequalities in view of $|A_0| \geq s$ and $|B_0| \geq t$ (which hold trivially in view of the definitions of $s$ and $t$). Moreover, if $r = 2$, then we can slightly improve this to $h_0' > (1 - \frac{r}{6}) |B_0| - \frac{3}{2}$ or $h_0 > (1 - \frac{r}{6}) |A_0| - \frac{3}{2}$. As a result, (88)--(92) imply that (93)--(95) hold with $x = 1 - \frac{r}{6}$ and $\epsilon = \frac{3}{2}$, and in the case $r = 2$, both (94) and (95) hold strictly. Thus (96) yields

$$|A + B| \geq |\tilde{A}| + \frac{19}{r} |\tilde{B}| - \frac{11r + 3}{6r},$$

with a strict inequality for $r = 2$. For $r = 2$, this gives $|A + B| > |\tilde{A}| + \frac{16}{7} |\tilde{B}| - 5$, and for $r \geq 3$ we instead have $|A + B| \geq |\tilde{A}| + \frac{16}{7} |\tilde{B}| - \frac{32}{7}$, both of which contradict (21). It remains to consider the cases when $s, t \in [d-1, d]$ with $r = \min\{s, t\} \leq 5$.

Suppose $s = t = d - 1$. Then (52) implies that $h_0 \geq \frac{1}{s} |A_0| - 1$ and (80) implies $|J_{B_0 \setminus B_0}| \geq \frac{1}{s} |B_0| - \frac{1}{s} h_0 - 1$, implying $h_0 + |J_{B_0 \setminus B_0}| \geq \frac{1}{s} |B_0| + \frac{2s-1}{s} |A_0| - \frac{2s-1}{s}$. We can apply this inequality to estimate the parenthetical quantities in (89) and (91) for each of the values $s = r \in [2, 5]$ individually, and estimate the parenthetical quantity in (88) by 0. Then, for $s = 2$, averaging 3 copies of (88) with 13 copies of the resulting bound in (89) and 11 copies of the resulting bound in (91) yields $|A + B| \geq \frac{49|A| + 53|B| - 138}{27} \geq |\tilde{A}| + \frac{25}{9} |\tilde{B}| - \frac{46}{9}$, contrary to (21) in view of $|\tilde{B}| \geq 2$. For $s = 3$, averaging 156 copies of (88) with 111 copies of the resulting bound in (89) and 105 copies of the resulting bound in (91) yields $|A + B| \geq \frac{687|A| + 705|B| - 1752}{27}$, contrary to (21). For $s = 4$, averaging 205 copies of (88) with 81 copies of the resulting bound in (89) and 79 copies of the resulting bound in (91) yields $|A + B| \geq \frac{681|A| + 689|B| - 1650}{365}$, contrary to (21). For $s = 5$,
averaging 546 copies of (88) with 151 copies of the resulting bound in (89) and 149 copies of the resulting bound in (91) yields \(|A + B| \geq \frac{1591|A|+1601|B|-3732}{846}\), contrary to (21).

Suppose \(s = r = d - 1\) and \(t = d\). Then (81) implies \(|J_{A_0} \setminus A_0| \geq \frac{1}{2}|A_0| - \frac{1}{s}h_0' - 1\) with \(t = s + 1\). We can apply this inequality to estimate the parenthetical quantities in (88), (90) and (92), yielding

\[
|A + B| \geq |A| + |B| - 3 + \frac{s + 1}{s}|A_0| + |B_0| - \frac{1}{s}h_0'.
\]

\[
|A + B| \geq |A| + (s + 1)|B| - (s + 3) + \frac{s + 1}{s}|A_0| - s|B_0| + \frac{s - 1}{s}h_0'.
\]

\[
|A + B| \geq (s + 2)|A| + |B| - (s + 4) - \frac{s^2 + s - 1}{s}|A_0| + |B_0| + \frac{s - 1}{s}h_0'.
\]

Averaging \(s^3 + s^2 - 2s - 1\) copies of the bound in (97) with \(s^2 + 2s\) copies of the bound in (98) and \(s^2 + 2s + 1\) copies of the bound in (99) yields

\[
|A + B| \geq \frac{(2s^3 + 6s^2 + 5s + 1)|A| + (2s^3 + 5s^2 + 2s)|B| - (5s^3 + 14s^2 + 9s + 1) + (s^2 + s - 1)b_0'}{s(s + 1)(s + 2)}.
\]

For the values \(s = 2, 3, 4, 5\), the above bound, using \(h_0' \geq 0\), becomes \(|A + B| \geq \frac{51|A|+40|B|-115}{24},\)

\(|A + B| \geq \frac{124|A|+105|B|-289}{60},\)

\(|A + B| \geq \frac{245|A|+216|B|-581}{120},\)

\(|A + B| \geq \frac{426|A|+385|B|-1021}{210},\)

respectively, all of which contradict (21).

If \(s = d\) and \(t = r = d - 1\), then (80) implies \(|J_{B_0} \setminus B_0| \geq \frac{1}{7}|B_0| - \frac{1}{s}h_0 - 1\). In view of the symmetry between the variables \(|A_0|\) and \(|B_0|\) in (88)–(92), we can then repeat the above calculation, swapping the roles of \(|A_0|\) and \(|B_0|\), of \(s\) and \(t\) and of \(h_0'\) and \(h_0\), and using (89) and (91) in place of (90) and (92), to yield the same contradiction as in the last paragraph. So it remains to consider the case when \(r = s = t = d \in [2, 5]\).

If \(\Omega_i \geq h_i^*\) for at least \(s - 1\) indices \(i \in [1, s]\), then (89), \(s = d\) and (50) imply that (★) holds, as desired. Thus

\[
h_i^* > 0
\]

and, since Claims 5 and 6 imply that \(h_i^* \leq |B_1| - 2\), we conclude that there must be distinct indices \(\alpha, \beta \in [1, s]\) with \(|A_0^\alpha| - 2 \leq \Omega_\alpha < h_1^* \leq |B_1| - 2\) and \(|A_0^\beta| - 2 \leq \Omega_\beta < h_1^* \leq |B_1| - 2\). Consequently, letting \(\Theta = \{\alpha, \beta\}\), we can apply (56) with \(|\Theta| = 2\) for \(s \geq 3\). Before using this estimate, let us first show that we must have \(\Omega_i < |A_0^\gamma| - 2\) for some \(\gamma \in [1, s]\).

If \(\Omega_i \geq |A_0^i| - 2\) for every \(i \in [1, s]\), then \(\sum_{i=1}^{s} \Omega_i \geq |A_0| - 2s\). Using this estimate along with (51) and (100) in (89) yields

\[
|A + B| \geq |A| + (s + 1)|B| - 3s + |A_0| - (s - 1)|B_0|.
\]

Using the estimates (51), (100) and \(t = s \in (91)\) yields

\[
|A + B| \geq (s + 1)|A| + |B| - s - (s + 1)|A_0| + 2|B_0|.
\]
Averaging $s^2 - 3$ copies of (88) (using (100) to estimate the quantity in parenthesis) with $s + 3$
s of copies of (101) and $s$ copies of (102) yields
\[ |A + B| \geq \frac{(2s + 2)|A| + (2s + 5)|B| - (5s + 9) + \frac{3}{s}}{s + 2} > |A| + \frac{(3s + 5)|\tilde{B}| - (5s + 9)}{s + 2}. \]

The derivative with respect to $s$ of the above expression is $\frac{\overline{B} - 1}{(s+2)^2} \geq 0$, meaning the bound will
be minimized for the minimal value of $s = d \geq 2$, implying $|A + B| > |A| + \frac{11|\tilde{B}| - 10}{4}$, contrary
to (21). So we may now assume there exists some $\gamma \in [1,s]$ with $\Omega_{\gamma} < |A_0|^2 - 2$.

From (88), we have
\[ |A + B| \geq |A| + |B| - 2 + |A_0| + |B_0| + h_1^\gamma. \]  
\[ (103) \]

Observe that $\gamma \in [1,s]$ must be distinct from the distinct indices $\alpha, \beta \in [1,s]$ with $\Omega_{\gamma} \geq h_1^\gamma$ (as
$h_1^\gamma \leq |B_1| - 2$ from Claims 5 and 6), which implies that $s \geq 3$. In view of $\Omega_{\gamma} \geq h_1^\gamma$, $h_0 \geq h_0^\gamma$ and
and (56) (applied with $|\Theta| = 2$), we have $h_0 + \sum_{i=1}^s \Omega_i \geq h_0^\gamma + (\Omega_\alpha + \Omega_\beta) + \Omega_{\gamma} \geq
\frac{2}{s-2}(|A_0| - 2h_1^\gamma - 2) - 6 + h_1^\gamma$. Applying this estimate in (89) yields
\[ |A + B| \geq |A| + (s + 1)|B| - \frac{s^2 + 6s - 12}{s - 2} + \frac{s}{s - 2}|A_0| - s|B_0| + \frac{2s - 8}{s - 2}h_1^\gamma. \]  
\[ (104) \]

Using the estimate (51) in (91) and recalling that $t = s$
gives
\[ |A + B| \geq (s + 1)|A| + |B| - (s + 1) - (s + 1)|A_0| + 2|B_0| + h_1^\gamma. \]  
\[ (105) \]

Averaging $(s^2 - s - 4)s$ copies of the bound in (103) with $s^2 + s - 6 = (s - 2)(s + 3)$ copies of the bound in (104) and $(s - 1)s$ copies of the bound in (105) yields $|A + B|$ being at least
\[ \frac{(2s^3 - 4s - 6)|A| + (2s^3 + 2s^2 - 10s - 6)|B| - (4s^3 + 7s^2 - 3s - 36) + (s^3 + 2s^2 - 7s - 24)h_1^\gamma}{s^3 + s^2 - 4s - 6} \]
\[ \geq |\tilde{A}| + \frac{(3s^3 + s^2 - 10s - 6)|\tilde{B}| - (4s^3 + 7s^2 - 3s - 36) + (s^3 + 2s^2 - 7s - 24)h_1^\gamma}{s^3 + s^2 - 4s - 6}. \]  
\[ (106) \]

The coefficient of $h_1^\gamma$ in the numerator of (106) is non-negative for $s \geq 3$. Thus, using the estimate
$h_1^\gamma \geq 1$ (from (100)), the bound in (106), for $s = 4, 5$, becomes $|A + B| \geq |\tilde{A}| + \frac{162|\tilde{B}| - 276}{58}$ and
$|A + B| \geq |\tilde{A}| + \frac{344|\tilde{B}| - 508}{124}$, respectively, both of which contradict (21). It remains to consider the case $s = t = d = 3$, for which (106) implies $|A + B| \geq 2|A| + 2|B| - 7$, which contradicts (21) unless $|A|, |B| \leq 7$. From (100) and Claims 6 and 5, we see that $1 \leq h_1^\gamma \leq |B_1| - 2$. Thus
$|B_1| \geq 3$, whence (50) and (100) imply $|P_B \setminus B| \geq |B_1| - 1 \geq 7 \geq |B|$, contrary to (25) and (23), which completes the proof. \[ \Box \]

4. Further Commentary and Lower Bound Examples

The examples below show that the bounds for $|P_A \setminus A|, |P_A \setminus A| + |P_B \setminus B| \text{ and } |P \setminus A| + |P \setminus B|$ in Theorem 1.3 are all tight. What is not tight in Theorem 1.3 is the hypothesis $|A + B| \leq |A| + \frac{19}{7} |B| - 5$, and we would expect the theorem to remain true under a weaker small subset
hypothesis, likely replacing $\frac{12}{7}$ with something closer to 3. It is also plausible that the bound for $|P_B \setminus B|$ could be improved when $r' < r$, as the examples below showing tightness for $|P_B \setminus B| \leq r$ all have $r = r'$.

**Example 1.** Let $k \geq 3$ and $r \in [0,k-3]$ be integers. Let
\[ A = ([1,k-2] \times \{0\}) \cup ([0,r+1] \times \{1\}) \quad \text{and} \quad B = ([0,k-3] \times \{0\}) \cup ([0,r+1] \times \{1\}). \]
Then $|A| = |B| = k$, $\delta(A,B) = 0$, $P_A = ([1,k-2] \times \{0\}) \cup ([0,r+1] \times \{1\})$, $P_B = ([0,k-3] \times \{0\}) \cup ([0,r+1] \times \{1\})$, $P = ([0,k-2] \times \{0\}) \cup ([0,r+1] \times \{1\})$ and $|A + B| = |A| + 2|B| - 2 + r \leq |A| + 3|B| - 5$. Also,
\[ |P_A \setminus A| = |P_B \setminus B| = r \quad \text{and} \quad |P \setminus A| + |P \setminus B| = 2r + 2 - 2r = |A| - |B|, \]
showing tightness in the bounds from Theorem 1.3.

**Example 2.** Let $a > b \geq 2$ be even integers and let
\[ A = ([0,a/2 - 1] \times \{0\}) \cup ([0,a/2 - 1] \times \{1\}) \quad \text{and} \quad B = ([0,b/2 - 1] \times \{0\}) \cup ([0,b/2 - 1] \times \{1\}). \]
Then $|A| = a$, $|B| = b$, $\delta(A,B) = 0$, $P_A = P = A$, $P_B = B$ and $|A + B| = |A| + |B| + \frac{|A| + |B|}{2} - 1$, so $r' = 0$ and $r = \frac{|A| - |B|}{2} - 1$. However,
\[ |P \setminus A| + |P \setminus B| = |A| - |B| = 2r + 2 = 2r + 2 - |A| - |B| + |P_A| - |P_B|, \]
showing the bound for $|P \setminus A| + |P \setminus B|$ can be tight and is not bounded as a function of $r'$.

**Example 3.** Let $a \geq b + 2 \geq 4$ and $r \in [a-b-2,a-3]$ be integers. Let
\[ A = ([0,a-3] \times \{0\}) \cup ([0,r+1] \times \{1\}) \quad \text{and} \quad B = ([0,b-2] \times \{0\}) \cup ([0,1] \times \{1\}). \]
Then $|A| = a$, $|B| = b$, $\delta(A,B) = 0$, $P_A = ([0,a-3] \times \{0\}) \cup ([0,r+1] \times \{1\})$, $P_B = ([0,b-2] \times \{0\}) \cup ([0,1] \times \{1\})$, and $P = ([0,a-3] \times \{0\}) \cup ([0,r+1] \times \{1\})$. Also,
\[ 2|A| + |B| - 4 \leq |A + B| = |A| + 2|B| - 2 + r \leq 2|A| + 2|B| - 5, \]
\[ |P_B \setminus B| = 0, \quad |P_A \setminus A| = r, \quad \text{and} \]
\[ |P \setminus A| + |P \setminus B| = 2r + |A| - |B| \leq 3r + 2 = 2r + 2 + |P_A| - |P_B| - |A| - |B|. \]
Furthermore, choosing $r = a - b - 2 = |A| - |B| - 2$, the inequality above becomes an equality, showing the bound $3r + 2$ can be tight in Theorem 1.3, and $|P_B \setminus B| + |P_A \setminus A| = |P_A \setminus A| = r = 2r'$, showing this bound to be tight as well.

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