REPRESENTING SEQUENCE SUBSUMS AS SUMSETS OF NEAR EQUAL SIZED SETS

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Abstract. For a sequence S of terms from an abelian group G of length $|S|$, let $\Sigma_n(S)$ denote the set of all elements that can be represented as the sum of terms in some n-term subsequence of S. When the subsum set is very small, $|\Sigma_n(S)| \leq |S| - n + 1$, it is known that the terms of S can be partitioned into n nonempty sets $A_1, \ldots, A_n \subseteq G$ such that $\Sigma_n(S) = A_1 + \ldots + A_n$. Moreover, if the upper bound is strict, then $|A_i \setminus Z| \leq 1$ for all i, where $Z = \bigcap_{i=1}^n (A_i + H)$ and $H = \{g \in G : \ g + \Sigma_n(S) = \Sigma_n(S)\}$ is the stabilizer of $\Sigma_n(S)$. This allows structural results for sumsets to be used to study the subsum set $\Sigma_n(S)$ and is one of the two main ways to derive the natural subsum analog of Kneser's Theorem for sumsets. In this paper, we show that such a partitioning can be achieved with sets A_i of as near equal a size as possible, so $\lfloor \frac{|S|}{n} \rfloor \leq |A_i| \leq \lceil \frac{|S|}{n} \rceil$ for all i, apart from one highly structured counterexample when $|\Sigma_n(S)| = |S| - n + 1$ with $n = 2$. The added information of knowing the sets A_i are of near equal size can be of use when applying the aforementioned partitioning result, or when applying sumset results to study $\Sigma_n(S)$ (e.g., [20]). We also give an extension increasing the flexibility of the aforementioned partitioning result and prove some stronger results when $n \geq \frac{1}{2}|S|$ is very large.

1. Introduction

Basic Notation. Let G be an abelian group. Following standard conventions in Combinatorial Number Theory (see [37] [22] [21]), by a sequence S of terms from G, we mean a finite, *unordered* string of elements

$$S = g_1 \cdot \ldots \cdot g_\ell$$

with $g_i \in G$ the terms of the sequence S, each term separated via the boldsymbol $\cdot$ (differentiating it from multiplication in circumstances where both operations are in use). Formally, a sequence is considered as an element of the free abelian monoid $\mathcal{F}(G)$ with basis G and operation $\cdot$, giving a standardized system of notation for sequences. Given an element $g \in G$, we let $\mathsf{v}_g(S) \geq 0$ denote the number of occurrences of the term g in S and let $g^{[n]}$ represent the sequence consisting of the element g repeated n times, so that any sequence $S \in \mathcal{F}(G)$ has the form

$$S = \prod_{g \in G}^{\bullet} g^{[\mathsf{v}_g(S)]}.$$

We let $T \mid S$ denote that T is a subsequence of S, so $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G$, and in such case use $T^{[-1]} \cdot S$ or $S \cdot T^{[-1]}$ to denote the sequence obtained by removing from S the terms in T, so $\mathsf{v}_g(T^{[-1]}\cdot S)=\mathsf{v}_g(S)-\mathsf{v}_g(T)$. The support of the sequence S is the set of all elements occurring in S:

$$\mathrm{Supp}(S)=\{g\in G:\ \mathsf{v}_g(S)>0\}.$$

For a subset $X\subseteq G$, let $S_X\mid S$ denote the subsequence of S consisting of all terms from X, so

$$S_X=\prod_{g\in X}^{\bullet} g^{[\mathsf{v}_g(S)]}.$$

Then $|S|=\ell$ is the length of the sequence,

$$\mathsf{h}(S)=\max\{\mathsf{v}_g(S):\ g\in G\}$$

is the maximum multiplicity of a term in S, $\sigma(S)=g_1+\ldots+g_n$ is the sum of S, and

$$\Sigma_n(S)=\{\sigma(T):\ T\mid S,\ |T|=n\}\subseteq G$$

is the set of n-term subsums of S, for $n\geq 0$.

All intervals are discrete, so $[m,n]=\{x\in\mathbb{Z}:\ m\leq x\leq n\}$. Given subsets $A_1,\ldots,A_n\subseteq G$, their sumset is defined as

$$A_1+\ldots+A_n=\{a_1+\ldots+a_n:\ a_i\in A_i\}.$$

The stabilizer of a set $A\subseteq G$ is the subgroup $\mathsf{H}(A)=\{g\in G:\ g+A=A\}\leq G$, which is the largest subgroup H such that A is a union of H-cosets. If $\mathsf{H}(A)$ is trivial, then A is *aperiodic*, and otherwise A is *periodic*. We say that A is H*-periodic* if A is a union of H-cosets, equivalently, if $H\leq\mathsf{H}(A)$. For $x\in G$ and $A,\ B\subseteq G$, we let $\mathsf{r}_{A+B}(x)=|(A-x)\cap B|=|\{(a,b)\in A\times B:\ a+b=x\}|$ denote the number of representations for x as an element of $A+B$, and call $x\in A+B$ a unique expression element when $\mathsf{r}_{A+B}(x)=1$. For $H\leq G$, we let $\phi_H:G\to G/H$ denote the natural homomorphism.

Background. The study of sequence subsums is a classical topic in Combinatorial Number Theory. Often, it is desired that either $0\in\Sigma_n(A)$, or $|\Sigma_n(S)|$ is large, or $\Sigma_n(S)=G$, and either conditions that guarantee the appropriate outcome, or the structure of sequences failing to satisfy the desired outcome, are sought. The Erdős-Ginzburg-Ziv Theorem [37] [45] [10] and Davenport Constant [37] [22] [43] are two such examples of very well-studied problems along these lines. A selection of other examples may be found here [1] [16] [24] [47] [48] [49].

One effective tool for studying $\Sigma_n(S)$, e.g., employed in the original proof of the Erdős-Ginzburg-Ziv Theorem [10], is via setpartitions. Consider a sequence $\mathcal{A}=A_1\cdot\ldots\cdot A_n$ whose terms A_i are *nonempty* (and finite) subsets of G. We call such a sequence a *setpartition* over G. Note the setpartition $\mathcal{A}$ naturally partitions the terms in its underlying sequence

$$\mathsf{S}(\mathcal{A}):=\prod_{i\in[1,n]}^{\bullet}\prod_{g\in A_i}^{\bullet} g$$

into n nonempty sets. It is then rather immediate that $\sum_{i=1}^{n} A_i \subseteq \Sigma_n(S)$ when $\mathsf{S}(\mathcal{A}) \mid S$, which allows sumset results to be used for studying $\Sigma_n(S)$. This becomes even more effective if we know there is some setpartition $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$ such that equality holds, $\sum_{i=1}^{n} A_i = \Sigma_n(S)$, for this means the subsums $\Sigma_n(A)$ can be represented as an ordinary sumset, and sumset results directly applied. The more structure that is known for the A_i, the easier and more effective it is to apply the corresponding sumset results. While this cannot hold for a general sequence, we have the striking fact that this is always possible so long as $|\Sigma_n(S)|$ is sufficiently small [37, Theorem 14.1] [26].

Theorem A (Partition Theorem)**.** *Let G be an abelian group, let $n \geq 1$, let $S \in \mathcal{F}(G)$ be a sequence of terms from G, and suppose $S' \mid S$ is a subsequence with $\mathsf{h}(S') \leq n \leq |S'|$. Then there exists a setpartition $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$ and $|\mathsf{S}(\mathcal{A})| = |S'|$ such that either*

1. $|\Sigma_n(S)| \geq |\sum_{i=1}^{n} A_i| \geq |S'| - n + 1$, *or*
2. $\Sigma_n(S) = \sum_{i=1}^{n} A_i$, $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \boldsymbol{\cdot} S) \subseteq Z$ *and* $|A_i \setminus Z| \leq 1$ *for all* i, *where* $Z = \bigcap_{i=1}^{n}(A_i + H)$ *and* $H = \mathsf{H}(\Sigma_n(S))$.

Theorem A ensures that $\Sigma_n(S) = \sum_{i=1}^{n} A_i$, for some setpartition $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$ and $|\mathsf{S}(\mathcal{A})| = |S'|$, provided $|\Sigma_n(S)| \leq |S'| - n + 1$, with additional structural information holding when the upper bound is strict. Worth noting, Theorem A can always be applied (so long as $|S| \geq n$) with S' taken to be the maximal subsequence of S with $\mathsf{h}(S') \leq n$. In case Theorem A.2 holds, this allows us to apply Kneser's Theorem [42] [37] [45] [22] to derive yet more information regarding $\Sigma_n(S)$, which is often incorporated into the statement of Theorem A itself (e.g. [37, Theorem 14.1] [38]).

Theorem B (Kneser's Theorem)**.** *Let G be an abelian group, let $A_1, \ldots, A_n \subseteq G$ be finite, nonempty subsets, and let $H = \mathsf{H}(\sum_{i=1}^{n} A_i)$. Then*

$$|\sum_{i=1}^{n} A_i| \geq \Big(\sum_{i=1}^{n} |\phi_H(A_i)| - n + 1\Big)|H| = \sum_{i=1}^{n} |A_i + H| - (n-1)|H|.$$

Kneser's Theorem is the fundamental lower bound for sumsets in an abelian group. Combining it with Theorem A (applied modulo H) yields the analogous result for sequence subsums [38].

Theorem C (Subsum Kneser's Theorem)**.** *Let G be an abelian group, let $n \geq 1$, let $S \in \mathcal{F}(G)$ be a sequence of terms from G with $|S| \geq n$, and let $H = \mathsf{H}(\Sigma_n(S))$. Then*

$$|\Sigma_n(S)| \geq \Big(|\phi_H(S')| - n + 1\Big)|H|,$$

where $S' \mid S$ is a maximum length subsequence with $\mathsf{h}(\phi_H(S')) \leq n$.

Note $|\phi_H(S')| = \sum_{g\in G/H} \max\{n,\, \mathsf{v}_g(\phi_H(S))\}$. The Subsum Kneser's Theorem can alternatively be derived as a special case of the DeVos-Goddyn-Mohar Theorem [9] [37]. Theorem C, and the more general Theorem A, have found numerous use in problems regarding sequence subsums [5] [15] [18] [19] [23] [25] [27] [28] [29] [30] [31] [32] [33] [34] [36], extending, complementing or resolving questions of established interest [2] [3] [4] [6] [7] [8] [10] [11] [12] [13] [14] [17] [39] [40] [41] [44] [46] [50].

In this paper, we will further strengthen Theorem A. Theorem 1.1 applies to the more general object $X + \Sigma_n(S)$ rather than $\Sigma_n(S)$ (which is the case $X = \{0\}$), showing that Theorem A holds even if a fixed portion is "frozen" in the set X. For instance, if $X \subseteq \Sigma_m(T)$, then $X + \Sigma_n(S) \subseteq \Sigma_{m+n}(T \cdot S)$, and we obtain the conclusion of Theorem A under the restriction of only being able to repartition the terms from S. Theorem 1.1 also shows that, apart from one highly structured counter-example characterized in Theorem 1.1.3, the resulting setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ can be chosen such that the sizes of the sets A_i are as near equal as possible, i.e., with $||A_i| - |A_j|| \leq 1$ for all $i, j \in [1, n]$ (equivalently, $\lfloor \frac{|\mathsf{S}(\mathcal{A})|}{n} \rfloor \leq |A_i| \leq \lceil \frac{|\mathsf{S}(\mathcal{A})|}{n} \rceil$ for all i). We call such a setpartition *equitable*. While such improvements are not needed for every application of Theorem A, they can simplify technical issues related to the use of Theorem A, sometimes in an essential fashion. For example, the results of this paper (Sections 2 and 3) are needed to prove the main result in the forthcoming paper [20] dealing with refined properties of product-one sequences over a dihedral group.

Theorem 1.1. *Let G be an abelian group, let $n \geq 1$, let $X \subseteq G$ be a finite, nonempty set, let $L \leq \mathsf{H}(X)$, let $S \in \mathcal{F}(G)$ be a sequence of terms from G, and suppose $S' \mid S$ is a subsequence with $\mathsf{h}(\phi_L(S')) \leq n \leq |S'|$. Then there is a setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$, $|\mathsf{S}(\mathcal{A})| = |S'|$ and $|\phi_L(A_i)| = |A_i|$ for all $i \in [1, n]$ such that*

1. $|X + \Sigma_n(S)| \geq |X + \sum_{i=1}^{n} A_i| \geq (|S'| - n)|L| + |X|$ *and $\mathcal{A}$ is equitable, or*
2. $X + \Sigma_n(S) = X + \sum_{i=1}^{n} A_i$, *$\mathcal{A}$ is equitable,* $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S) \subseteq Z$ *and $|A_i \setminus Z| \leq 1$ for all i, where $Z = \bigcap_{i=1}^{n}(A_i + H)$ and $H = \mathsf{H}(X + \Sigma_n(S))$, or*
3. *$n = 2$, $X \setminus (\beta + L)$ and $(A_1 + L) \cap (A_2 + L)$ are K-periodic,* $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S) \subseteq (A_1 + L) \cap (A_2 + L)$, *$\big((A_1 + L) \cup (A_2 + L)\big) \setminus \big((A_1 + L) \cap (A_2 + L)\big)$ is a K-coset,* $\mathsf{H}(X + \Sigma_n(S)) = \mathsf{H}(X + \sum_{i=1}^{n} A_i) = \mathsf{H}(X) = L$, *and $|X + \Sigma_n(S)| = (|S'| - n)|L| + |X|$, for some $\beta \in X$ and $K \leq G$ with $L \leq K$ and $K/L \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

In Section 3, we will also derive some additional strengthenings of Theorem 1.1 in the case $n \geq \frac{1}{2}|S'|$ is very large. In particular, we will achieve the same strengthened conclusions recently guaranteed in [38] under a different n is large assumption (Theorem 3.2). This, in turn, will allow us to derive additional information for S, in particular, when $|S| = 2n$ with $|\Sigma_n(S)| \leq n+1$ and $\mathsf{h}(S) \leq n$ (Theorem 3.3).

2. Partitioning Results for General n

In this section, we will make heavy use of the arguments used to prove [37, Theorem 14.1] and the following easy consequence of Kneser's Theorem (see [37, Theorem 5.1]).

Theorem D. *Let G be an abelian group, and let A, $B \subseteq G$ be finite, nonempty subsets. If $|A+B| < |A|+|B|-1$, then $A+(B\setminus\{x\}) = A+B$ for all $x \in B$.*

We will also need the following observation, that follows by a routine induction on n.

Lemma 2.1. *Let G be an abelian group and let $A_1,\ldots,A_n \subseteq G$ be finite, nonempty subsets. Suppose $|\sum\limits_{i=1}^{j} A_i| \geq |\sum\limits_{i=1}^{j-1} A_i| + |A_j| - 1$ for all $j \in [2,n]$. Then $|\sum\limits_{i=1}^{n} A_i| \geq \sum\limits_{i=1}^{n}|A_i| - n + 1$. Moreover, if $|\sum\limits_{i=1}^{n} A_i| = \sum\limits_{i=1}^{n}|A_i| - n + 1$, then $|\sum\limits_{i=1}^{j} A_i| = |\sum\limits_{i=1}^{j-1} A_i| + |A_j| - 1$ for all $j \in [2,n]$.*

Let G be an abelian group, let $X \subseteq G$ be a nonempty subset and let $S \in \mathcal{F}(G)$ be a sequence. A setpartition $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$ will be called *maximal relative to* X if any setpartition $\mathcal{B} = B_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} B_n$ with $\mathsf{S}(\mathcal{B}) \mid S$, $|\mathsf{S}(\mathcal{B})| = |\mathsf{S}(\mathcal{A})|$ and $X + \sum\limits_{i=1}^{n} A_i \subseteq X + \sum\limits_{i=1}^{n} B_i$ has $X + \sum\limits_{i=1}^{n} A_i = X + \sum\limits_{i=1}^{n} B_i$. We simply say $\mathcal{A}$ is maximal relative to X if this is the case with $S = \mathsf{S}(\mathcal{A})$.

Lemma 2.2. *Let G be an abelian group, let $n \geq 1$, let $X \subseteq G$ be a finite, nonempty subset, let $S \in \mathcal{F}(G)$ be a sequence, let $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ be a setpartition with $\mathsf{S}(\mathcal{A}) \mid S$ maximal relative to X, and let $H = \mathsf{H}(X + \sum\limits_{i=1}^{n} A_i)$. Suppose*

$$|X + \sum_{i=1}^{n} A_i| < |X| + \sum_{i=1}^{n} |A_i| - n. \tag{1}$$

Then there exists a setpartition $\mathcal{B} = B_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} B_n$ with $\mathsf{S}(\mathcal{B}) \mid S$, $|\mathsf{S}(\mathcal{B})| = |\mathsf{S}(\mathcal{A})|$ and $X + \sum\limits_{i=1}^{n} B_i = X + \sum\limits_{i=1}^{n} A_i$ such that $\mathrm{Supp}(\mathsf{S}(\mathcal{B})^{[-1]} \boldsymbol{\cdot} S) \subseteq Z$ *and $|(y+H) \cap B_i| \leq 1$ for all $y \in G \setminus Z$ and $i \in [1,n]$, where $Z = \bigcap_{i=1}^{n}(B_i + H)$.*

Proof. In view of (1) and Kneser's Theorem, we conclude that H is nontrivial. Consider an arbitrary setpartition $\mathcal{B} = B_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} B_n$ with $\mathsf{S}(\mathcal{B}) \mid S$, $|S(\mathcal{B})| = |\mathsf{S}(\mathcal{A})|$ and $X + \sum\limits_{i=1}^{n} A_i \subseteq X + \sum\limits_{i=1}^{n} B_i$. Then $X + \sum\limits_{i=1}^{n} B_i = X + \sum\limits_{i=1}^{n} A_i$ since $\mathcal{S}(\mathcal{A}) \mid S$ with $\mathcal{A}$ maximal relative to X. Let $Z = \bigcap_{i=1}^{n}(B_i + H)$. In view of (1) and Lemma 2.1, there must be some $j \in [1,n]$ such that

$|X+\sum_{i=1}^{j}B_i|<|X+\sum_{i=1}^{j-1}B_i|+|B_j|-1$, in which case Theorem D implies

$$X+\sum_{i=1}^{j-1}B_i+(B_j\setminus\{x\})=X+\sum_{i=1}^{j}B_i\quad\text{ for all } x\in B_j. \tag{2}$$

Since $|X+\sum_{i=1}^{j}B_i|<|X+\sum_{i=1}^{j-1}B_i|+|B_j|-1$, Kneser's Theorem implies that $|(x+H')\cap B_j|\geq 2$ for every $x\in B_j$, where $H'=\mathsf{H}(X+\sum_{i=1}^{j}B_i)\leq H$. In particular,

$$|(x+H)\cap B_j|\geq 2\quad\text{ for all } x\in B_j. \tag{3}$$

Now further restrict $\mathcal{B}$ by assuming $\sum_{i=1}^{n}|\phi_H(B_i)|$ is maximal (subject to the defining condition for $\mathcal{B}$). Then we must have $B_j\subseteq Z$, where $j\in[1,n]$ is the index defined above. Indeed, if this fails, then there is some $x\in B_j\setminus Z$, and thus also some $k\in[1,n]$ with $\phi_H(x)\notin\phi_H(B_k)$ by definition of Z. We can then remove x from B_j and place it in B_k to yield a new setpartition $\mathcal{B}'=B_1'\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}B_n'$, where $B_j'=B_j\setminus\{x\}$, $B_k'=B_k\cup\{x\}$ and $B_i'=B_i$ for $i\neq j,k$. In view of (2), we have $X+\sum_{i=1}^{n}A_i=X+\sum_{i=1}^{n}B_i\subseteq X+\sum_{i=1}^{n}B_i'$, while in view of (3) and $\phi_H(x)\notin\phi_H(B_k)$, we have $\sum_{i=1}^{n}|\phi_H(B_i')|=\sum_{i=1}^{n}|\phi_H(B_i)|+1$, and now $\mathcal{B}'$ contradicts the maximality of $\sum_{i=1}^{n}|\phi_H(B_i)|$ for $\mathcal{B}$. Therefore

$$B_j\subseteq Z. \tag{4}$$

Claim A. $|(y+H)\cap B_i|\leq 1$ for all $y\in G\setminus Z$ and $i\in[1,n]$.

Proof. Assume by contradiction there is some $k\in[1,n]$ and $y\in B_k\setminus Z$ with $|(y+H)\cap B_k|\geq 2$. Let $\mathcal{C}=C_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}C_n$ be a setpartition with $\mathsf{S}(\mathcal{C})=\mathsf{S}(\mathcal{B})$, $X+\sum_{i=1}^{n}C_i=X+\sum_{i=1}^{n}A_i$, and $C_i\setminus Z=B_i\setminus Z$ and $\phi_H(C_i)=\phi_H(B_i)$ for all i, such that $|C_k|$ is maximal. Since $\phi_H(C_i)=\phi_H(B_i)$ for all i, we still have $\sum_{i=1}^{n}|\phi_H(C_i)|$ maximal, while $Z=\bigcap_{i=1}^{n}(B_i+H)=\bigcap_{i=1}^{n}(C_i+H)$. Thus, let $j'\in[1,n]$ be an index so that $C_{j'}$ satisfies (2), (3) and (4) for $\mathcal{C}$ (in place of B_j).

Suppose $C_{j'}\subseteq C_k$. Since $y\notin Z$ but $C_{j'}\subseteq Z$ (by (4)), we actually have $C_{j'}\subseteq C_k\setminus\{y\}$, in which case

$$C_{j'}+C_k\subseteq(C_{j'}\cup\{y\})+(C_k\setminus\{y\}). \tag{5}$$

In such case, we can define a new setpartition $\mathcal{C}'=C_1'\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}C_n'$ by removing y from C_k and placing it in $C_{j'}$, so $C_k'=C_k\setminus\{y\}$, $C_{j'}'=C_{j'}\cup\{y\}$ and $C_i'=C_i$ for $i\neq k,j'$. Note $y\in B_k\setminus Z=C_k\setminus Z$. In view of (5), we have $X+\sum_{i=1}^{n}A_i=X+\sum_{i=1}^{n}C_i\subseteq X+\sum_{i=1}^{n}C_i'$, while in view of $|(y+H)\cap C_k|=|(y+H)\cap B_k|\geq 2$ (as $y\notin Z$ and $C_k\setminus Z=B_k\setminus Z$) and $\phi_H(y)\notin\phi_H(C_{j'})$ (as

$y \notin Z$ and $C_{j'} \subseteq Z$ by (4)), we have $\sum_{i=1}^{n}|\phi_H(C'_i)| = \sum_{i=1}^{n}|\phi_H(C_i)|+1 = \sum_{i=1}^{n}|\phi_H(B_i)|+1$, so that $\mathcal{C}'$ contradicts the maximality of $\sum_{i=1}^{n}|\phi_H(B_i)|$ for $\mathcal{B}$. So we instead conclude that $C_{j'} \nsubseteq C_k$. Thus, in view (4), it follows that there is some $x \in C_{j'} \subseteq Z$ with $x \notin C_k$.

In this case, we define a new setpartition $\mathcal{C}' = C'_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} C'_n$ by removing x from $C_{j'}$ and placing it in C_k, so $C'_{j'} = C_{j'} \setminus \{x\}$, $C'_k = C_k \cup \{x\}$ and $C'_i = C_i$ for $i \neq j', k$. In view of (2), we have $X + \sum_{i=1}^{n} A_i = X + \sum_{i=1}^{n} C_i \subseteq X + \sum_{i=1}^{n} C'_i$, while in view of (3) and $x \in Z$, we have $\phi_H(C'_i) = \phi_H(C_i) = \phi_H(B_i)$ and $C'_i \setminus Z = C_i \setminus Z = B_i \setminus Z$ for all i. Thus, since $|C'_k| = |C_k|+1$, we see that $\mathcal{C}'$ contradicts the maximality of $|C_k|$ for $\mathcal{C}$, completing the claim. □

In view of Claim A, we see that the lemma holds with the setpartition $\mathcal{B}$ unless there is some $y \in \mathrm{Supp}(\mathsf{S}(\mathcal{B})^{[-1]} \boldsymbol{\cdot} S)$ with $y \notin Z$. However, if this were the case, then $\phi_H(y) \notin \phi_H(B_j)$ in view of (4). Define a new setpartition $\mathcal{B}' = B'_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} B'_n$ by removing any term $x \in B_j$ from B_j and placing y into B_j instead, so $B'_j = B_j \setminus \{x\} \cup \{y\}$ and $B'_i = B_i$ for $i \neq j$. In view of (2), we have $X + \sum_{i=1}^{n} A_i = X + \sum_{i=1}^{n} B_i \subseteq X + \sum_{i=1}^{n} B'_i$, while in view of (3) and $\phi_H(y) \notin \phi_H(B_j)$, we have $\sum_{i=1}^{n}|\phi_H(B'_i)| = \sum_{i=1}^{n}|\phi_H(B_i)|+1$, in which case $\mathcal{B}'$ contradicts the maximality of $\sum_{i=1}^{n}|\phi_H(B_i)|$ for $\mathcal{B}$, completing the proof. □

Lemma 2.3. *Let G be an abelian group, let $n \geq 1$, let $X \subseteq G$ be a finite, nonempty subset, let $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ be a setpartition over G maximal relative to X, let $H = \mathsf{H}(X + \sum_{i=1}^{n} A_i)$ and let $Z = \bigcap_{i=1}^{n}(A_i + H)$. Suppose $|(y+H) \cap A_i| \leq 1$ for all $y \in G \setminus Z$ and $i \in [1,n]$, and*

$$|X + \sum_{i=1}^{n} A_i| < |X| + \sum_{i=1}^{n}|A_i| - n + (|H|-1). \tag{6}$$

Then there exists a setpartition $\mathcal{B} = B_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} B_n$ with $\mathsf{S}(\mathcal{B}) = \mathsf{S}(\mathcal{A})$, $X + \sum_{i=1}^{n} B_i = X + \sum_{i=1}^{n} A_i$ and $Z \subseteq \bigcap_{i=1}^{n}(B_i + H)$ such that $|B_i \setminus Z| \leq 1$ for all i.

Proof. In view of (6) and Kneser's Theorem, we conclude that H is nontrivial. Consider a setpartition $\mathcal{B} = B_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} B_n$ with

$$\begin{aligned} &\mathsf{S}(\mathcal{B}) = \mathsf{S}(\mathcal{A}), \quad X + \sum_{i=1}^{n} B_i = X + \sum_{i=1}^{n} A_i, \quad Z = \bigcap_{i=1}^{n}(A_i + H) \subseteq \bigcap_{i=1}^{n}(B_i + H) \quad \text{and} \\ &|(y+H) \cap B_i| \leq 1 \quad \text{for all } y \in G \setminus Z \text{ and } i \in [1,n]. \end{aligned} \tag{7}$$

Since $\mathcal{A}$ satisfies these conditions, it follows that such a setpartition $\mathcal{B}$ exists. Let $e = \sum_{i=1}^{n}|B_i \setminus Z| \geq 0$. Let $I_e \subseteq [1,n]$ be all those indices $i \in [1,n]$ with $B_i \setminus Z$ nonempty, and let $I_Z \subseteq [1,n]$ be all those indices $i \in [1,n]$ with $B_i \subseteq Z$. By re-indexing the B_i, we can w.l.o.g. assume $I_Z = [1,m]$ and $I_e = [m+1,n]$.

Suppose $|X + \sum_{i=1}^{j} B_i| \geq |X + \sum_{i=1}^{j-1} B_i| + |B_j| - 1$ for all $j \in [1,m]$. Then Lemma 2.1 implies $|X + \sum_{i=1}^{m} B_i| \geq |X| + \sum_{i=1}^{m} |B_i| - m$. Kneser's Theorem implies

$$|\Big(X + \sum_{i=1}^{m} B_i\Big) + \sum_{i=m+1}^{n} B_i| \geq |X + \sum_{i=1}^{m} B_i| + \sum_{i=m+1}^{n} |B_i + H| - (n-m)|H| \tag{8}$$
$$\geq |X + \sum_{i=1}^{m} B_i| + \sum_{i=m+1}^{n} |B_i| + e(|H|-1) - (n-m)|H|,$$

with the second inequality in view of the final condition in (7). Combined with the previous estimate for $|X + \sum_{i=1}^{m} B_i|$, we find

$$|X + \sum_{i=1}^{n} B_i| \geq |X| + \sum_{i=1}^{n} |B_i| - n|H| + (m+e)(|H|-1). \tag{9}$$

Note that $e \geq |I_e| = n - m$. If equality holds, then $|B_i \setminus Z| \leq 1$ follows for all i, completing the proof. Therefore we can instead assume $e \geq n - m + 1$, which combined with (9) yields $|X + \sum_{i=1}^{n} A_i| = |X + \sum_{i=1}^{n} B_i| \geq |X| + \sum_{i=1}^{n} |B_i| - n + (|H|-1) = |X| + \sum_{i=1}^{n} |A_i| - n + (|H|-1)$, contrary to hypothesis. So we may instead assume there is some $j \in [1,m]$ with $|X + \sum_{i=1}^{j} B_i| < |X + \sum_{i=1}^{j-1} B_i| + |B_j| - 1$. In particular, this argument shows that I_Z is nonempty for any setpartition satisfying (7), and Theorem D ensures that

$$X + \sum_{i=1}^{j-1} B_i + (B_j \setminus \{x\}) = X + \sum_{i=1}^{j} B_i \quad \text{for all } x \in B_j. \tag{10}$$

In particular, $|B_j| \geq 2$. Let $K = \mathsf{H}(X + \sum_{i=1}^{j} B_i) \leq H$. If $|(y+H) \cap B_j| = 1$ for some $y \in G$, then $|(y+K) \cap B_j| = 1$ as well, whence Kneser's Theorem implies $|\Big(X + \sum_{i=1}^{j-1} B_i\Big) + B_j| \geq |X + \sum_{i=1}^{j-1} B_i| + |B_j + K| - |K| \geq |X + \sum_{i=1}^{j-1} B_i| + |B_j| - 1$, contrary to the definition of j. Therefore we instead conclude that

$$|(y+H) \cap B_j| \geq 2 \quad \text{for all } y \in B_j. \tag{11}$$

Now assume our setpartition $\mathcal{B}$ satisfying (7) is chosen such that

M1. $|I_e|$ is maximal (subject to (7)), M2. $\sum_{i\in I_e}|B_i|$ is maximal (subject to (7) and M1).

If $|B_i\setminus Z|=1$ for every $i\in I_e=[m+1,n]$, then the setpartition $\mathcal{B}$ satisfies the conditions of the lemma. Therefore we may assume there is some $k\in I_e=[m+1,n]$ with distinct $y_1,y_2\in B_k\setminus Z$. Since $j\in[1,m]=I_Z$, we have $B_j\subseteq Z$. Thus $y_1,y_2\notin B_j$.

Suppose $B_s\subseteq B_k$ for some $s\in I_Z$. Then $B_s\subseteq B_k\setminus\{y_1\}$ and $B_s+B_k\subseteq(B_s\cup\{y_1\})+(B_k\setminus\{y_1\})$, the former as $y_1\notin Z$ but $B_s\subseteq Z$ as $s\in I_Z$. In such case, define a new setpartition $\mathcal{B}'=B'_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}B'_n$ by setting $B'_s=B_s\cup\{y_1\}$, $B'_k=B_k\setminus\{y_1\}$ and $B'_i=B_i$ for $i\neq s,k$. Then $B_s+B_k\subseteq(B_s\cup\{y_1\})+(B_k\setminus\{y_1\})=B'_s+B'_k$ ensures that $X+\sum_{i=1}^{n}A_i=X+\sum_{i=1}^{n}B_i\subseteq X+\sum_{i=1}^{n}B'_i$, and equality must hold as $\mathcal{A}$ is maximal relative to X. By definition, $\mathsf{S}(\mathcal{B}')=\mathsf{S}(\mathcal{B})=\mathsf{S}(\mathcal{A})$. Since $B_s\subseteq Z$, we have still have $|(y+H)\cap B'_i|\leq 1$ for all i and $y\in G\setminus Z$, while $Z\subseteq\bigcap_{i=1}^{n}B'_i$ follows since $y_1\notin Z$. Thus $\mathcal{B}'$ satisfies (7). Since $y_1\notin Z$ and $y_2\in B'_k\setminus Z$, we see $I_e\cup\{j\}\subseteq[1,n]$ is the subset of indices $i\in[1,n]$ for which $B'_i\setminus Z$ is nonempty, meaning $\mathcal{B}'$ contradicts the maximality condition M1 for $\mathcal{B}$. So we instead assume $B_s\nsubseteq B_k$ for all $s\in I_Z$. In particular, $B_j\nsubseteq B_k$, meaning there is some $x\in B_j\setminus B_k$.

In this case, define a new setpartition $\mathcal{B}'=B'_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}B'_n$ by setting $B'_j=B_j\setminus\{x\}$, $B'_k=B_k\cup\{x\}$ and $B'_i=B_i$ for $i\neq j,k$. In view of (10), we have $X+\sum_{i=1}^{n}A_i=X+\sum_{i=1}^{n}B_i\subseteq X+\sum_{i=1}^{n}B'_i$, and equality must hold as $\mathcal{A}$ is maximal relative to X. By definition, $\mathsf{S}(\mathcal{B}')=\mathsf{S}(\mathcal{B})=\mathsf{S}(\mathcal{A})$. Since $x\in B_j\subseteq Z$, we have still have $|(y+H)\cap B'_i|\leq 1$ for all i and $y\in G\setminus Z$, while $Z\subseteq\bigcap_{i=1}^{n}B'_i$ follows in view of (11). Thus $\mathcal{B}'$ satisfies (7). Since $x\in B_j\subseteq Z$, we see $I_e\subseteq[1,n]$ is still the subset of indices $i\in[1,n]$ for which $B'_i\setminus Z$ is nonempty, meaning $\mathcal{B}'$ satisfies M1. However, since $|B'_k|=|B_k|+1$, $k\in I_e$ and $j\notin I_e$, the maximality of $\sum_{i\in I_e}|B_i|$ for $\mathcal{B}$ is contradicted by $\mathcal{B}'$. □

Lemma 2.4. *Let G be an abelian group, let $n\geq 0$, let $X\subseteq G$ be a finite, nonempty subset, let $S\in\mathcal{F}(G)$ be a sequence, and let $\mathcal{A}=A_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}A_n$ be a setpartition with $\mathsf{S}(\mathcal{A})\mid S$, $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]}\boldsymbol{\cdot}S)\subseteq Z$, and $|A_i\setminus Z|\leq 1$ for all i, where $Z=Z+H\subseteq\bigcap_{i=1}^{n}(A_i+H)$ and $H\leq\mathsf{H}(X+\sum_{i=1}^{n}A_i)$. Then the following hold.*

1. $X+\Sigma_n(S)=X+\sum_{i=1}^{n}A_i$.
2. *If $Z=g+H$ for some $g\in G$, then $X+\Sigma_\ell(S)=X+\sum_{i=1}^{n}A_i+(\ell-n)g$ for any $\ell\in[n,n+|\mathsf{S}(\mathcal{A})^{[-1]}\boldsymbol{\cdot}S|]$.*

Proof. 1. Note $X+\sum_{i=1}^{n}A_i\subseteq X+\Sigma_n(S)$ holds trivially. Let $T\mid S$ with $|T|=n$ be arbitrary. Since $X+\sum_{i=1}^{n}A_i$ is H-periodic by hypothesis, to establish the reverse inclusion, it suffices to show $\sigma(\phi_H(T))\in\sum_{i=1}^{n}\phi_H(A_i)$. Write $T=x_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}x_s\boldsymbol{\cdot}y_{s+1}\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}y_n$, with the x_i the terms of T with $x_i\in G\setminus Z$, and the y_i the terms of T with $y_i\in Z$. In view of $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]}\boldsymbol{\cdot}S)\subseteq Z$

and $|A_i \setminus Z| \leq 1$ for all i, we can re-index the A_i so that $x_i \in A_i$ for $i \in [1,s]$. But now, since $y_j \in Z = Z + H \subseteq \bigcap_{i=1}^n (A_i + H) \subseteq A_j + H$ for all $j \geq s+1$, it follows that $\sigma(\phi_H(T)) = \phi_H(x_1) + \ldots + \phi_H(x_s) + \phi_H(y_{s+1}) + \ldots + \phi_H(y_n) \in \sum_{i=1}^n \phi_H(A_i)$, completing Item 1.

2. By translating all terms of S appropriately by $-g$, we can w.l.o.g. assume $g = 0$, whence $H \subseteq \bigcap_{i=1}^n (A_i + H)$. In particular, $H \cap A_i \neq \emptyset$ and $|A_i \setminus H| \leq 1$ for all i. Since $X + \sum_{i=1}^n A_i$ is H-periodic with $\mathsf{S}(\mathcal{A}) \mid S$, $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \boldsymbol{\cdot} S) \subseteq g + H = H$ and $n \leq \ell \leq n + |\mathsf{S}(\mathcal{A})^{[-1]} \boldsymbol{\cdot} S|$, we trivially have $X + \sum_{i=1}^n A_i = X + \sum_{i=1}^n A_i + (\ell - n)g \subseteq X + \Sigma_\ell(S)$. To show the reverse inclusion, let $T = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell$ be an arbitrary ℓ-term subsequence of S. Since $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \boldsymbol{\cdot} S) \subseteq g + H = H$ and $|A_i \setminus H| \leq 1$ and for all i, there are at most n non-zero terms in $\phi_H(T)$, and by re-indexing, we can w.l.o.g. assume $\phi_H(g_i) = 0$ for $i > n$. Then, since $H \cap A_i \neq \emptyset$, $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \boldsymbol{\cdot} S) \subseteq H$ and $|A_i \setminus H| \leq 1$ for all i, it follows that $\phi_H(g_1) + \ldots + \phi_H(g_\ell) = \phi_H(g_1) + \ldots + \phi_H(g_n) \in \sum_{i=1}^n \phi_H(A_i)$. Hence, since $X + \sum_{i=1}^n A_i$ is H-periodic, we conclude that $X + \sigma(T) = X + g_1 + \ldots + g_\ell \subseteq X + \sum_{i=1}^n A_i$. Since T was an arbitrary ℓ-term subsequence of S, this establishes the reverse inclusion $X + \Sigma_\ell(S) \subseteq X + \sum_{i=1}^n A_i$. □

Let G be an abelian group and $A \subseteq G$ a subset. We say A is *quasi-periodic* if there is a subset $A_\emptyset \subseteq A$ such that $A \setminus A_\emptyset$ is nonempty and periodic with $A_\emptyset$ contained in a $\mathsf{H}(A \setminus A_\emptyset)$-coset. If $H \leq G$ is a nontrivial subgroup, then an *H-quasi-periodic decomposition* is a partition $A = (A \setminus A_\emptyset) \cup A_\emptyset$ with $A_\emptyset$ a subset of an H-coset and $A \setminus A_\emptyset$ H-periodic (or empty). It is *reduced* if $A_\emptyset$ is not quasi-periodic. As is easily derived,

$$\mathsf{H}(A) = \mathsf{H}(A_\emptyset) \leq H \quad \text{when } \emptyset \neq A_\emptyset \subset A_\emptyset + H. \tag{12}$$

If $A_\emptyset$ is quasi-periodic, as exhibited by $A'_\emptyset \subseteq A_\emptyset$, then $A = (A \setminus A'_\emptyset) \cup A'_\emptyset$ is a quasi-periodic decomposition with $A'_\emptyset \subset A_\emptyset$. Every finite set $A \subseteq G$ has a reduced quasi-periodic decomposition, and this decomposition is unique unless $A \cup \{\alpha\}$ is periodic for some $\alpha \notin A$ (see [35, Proposition 2.1]). The Kemperman Structure Theorem [37, Theorem 9.1] implies that, if $A, B \subseteq G$ are finite, nonempty subsets with $|A+B| = |A| + |B| - 1$ and either $A + B$ aperiodic or containing a unique expression element, then there are H-quasi-periodic decompositions $A = (A \setminus A_\emptyset) \cup A_\emptyset$, $B = (B \setminus B_\emptyset) \cup B_\emptyset$ and $A + B = \Big((A+B) \setminus (A_\emptyset + B_\emptyset)\Big) \cup (A_\emptyset + B_\emptyset)$ with the pair $(A_\emptyset, B_\emptyset)$ satisfying one of four possible structural types (I)–(IV), each with explicitly defined restrictions on where, and how many, unique expression elements there are. We will make use of this theory, referencing the details regarding Kemperman's Critical Pair Theory rather than repeating the rather lengthy statements and details here.

Lemma 2.5. *Let G be an abelian group, let $H \le G$ be a subgroup with $|H| \ge 3$, and let $Y \subseteq G$ be a finite subset such that $Y \setminus \{y_0\}$ is H-periodic (or empty) for some $y_0 \in Y$.*

1. *Y is aperiodic with $Y = (Y \setminus \{y_0\}) \cup \{y_0\}$ its unique reduced quasi-periodic decomposition.*
2. *If there is a K-quasi-periodic decomposition $Y = Y_1 \cup Y_0$, then $y_0 \in Y_0$ and $Y_0 \setminus \{y_0\}$ is H-periodic. Moreover, if $|Y_0| \ge 2$, then $H \le K$.*
3. *If $A, B \subseteq G$ with $A + B = Y$ and $|A + B| = |A| + |B| - 1$, then there are $a_0 \in A$ and $b_0 \in B$ such that $A \setminus \{a_0\}$ and $B \setminus \{b_0\}$ are H-periodic with $a_0 + b_0 = y_0$.*

Proof. We may w.l.o.g. assume $H = \mathsf{H}(Y \setminus \{y_0\})$. We may also assume $Y \setminus \{y_0\}$ is nonempty, else Items 1–3 all hold trivially. Item 1 follows from [35, Proposition 2.1] and [35, Comment c.6].

2. Suppose $Y = Y_1 \cup Y_0$ is a K-quasi-periodic decomposition and let $Y_0 = (Y_0 \setminus Y_\emptyset) \cup Y_\emptyset$ be a reduced quasi-periodic decomposition of Y_0. Then $(Y \setminus Y_\emptyset) \cup Y_\emptyset$ is a reduced quasi-periodic decomposition of Y with either $\mathsf{H}(Y \setminus Y_\emptyset) = \mathsf{H}(Y_0 \setminus Y_\emptyset) \le \langle Y_0 - Y_0 \rangle \le K$ or $Y_0 = Y_\emptyset$ (by (12)). Hence Item 1 ensures that $Y_\emptyset = \{y_0\}$ with $H = \mathsf{H}(Y \setminus \{y_0\}) = \mathsf{H}(Y \setminus Y_\emptyset) \le K$ if $|Y_0| \ge 2$. If $Y_0 = \{y_0\} = Y_\emptyset$, then $Y_1 = Y \setminus \{y_0\} = Y \setminus Y_\emptyset$ is H-periodic. Otherwise, $H \le K$ ensures Y_1 is H-periodic, while $Y \setminus Y_\emptyset = Y \setminus \{y_0\}$ is H-periodic by hypothesis. In either case, Y_1 and $Y \setminus Y_\emptyset$ are both H-periodic, and it follows that $(Y \setminus Y_\emptyset) \setminus Y_1 = Y_0 \setminus Y_\emptyset = Y_0 \setminus \{y_0\}$ is also H-periodic. Item 2 now follows.

3. Suppose $A + B = Y$ and $|A + B| = |A| + |B| - 1$. Since $Y = A + B$ is aperiodic by Item 1, we can directly apply the Kemperman Structure Theorem [35, Proposition 2.1] to $A+B$ yielding associated K-quasi-periodic decompositions $A = (A \setminus A_\emptyset) \cup A_\emptyset$, $B = (B \setminus B_\emptyset) \cup B_\emptyset$ and $Y = (Y \setminus (A_\emptyset + B_\emptyset)) \cup (A_\emptyset + B_\emptyset)$. If the pair $(A_\emptyset, B_\emptyset)$ has type (IV), then $Y \cup \{\beta\}$ is periodic for some $\beta \in G \setminus Y$. In such case, any reduced quasi-periodic decomposition $Y = Y_1 \cup Y_0$ must have $|Y_0| \ge 2$ or $|\mathsf{H}(Y_1)| = 2$ (cf. [35, Proposition 2.1]), contrary to Item 1. If the pair $(A_\emptyset, B_\emptyset)$ has type (III), then Y is periodic, contrary to Item 1. If the pair $(A_\emptyset, B_\emptyset)$ has type (II), then either $Y \cup \{\beta\}$ is periodic for some $\beta \in G$, yielding the same contradiction as before, or else $Y = (Y \setminus (A_\emptyset + B_\emptyset)) \cup (A_\emptyset + B_\emptyset)$ is a reduced quasi-periodic decomposition of Y (by [35, Comment c.3]) with $|A_\emptyset + B_\emptyset| \ge 3$, again contrary to Item 1. We are left to conclude that $(A_\emptyset, B_\emptyset)$ has type (I), so w.l.o.g. $|A_\emptyset| = 1$, say with $A_\emptyset = \{a_0\}$. Hence $A + B = Y$ is a union of $a_0 + B$ with a K-periodic set. In particular, $(Y \setminus (a_0 + B_\emptyset)) \cup (a_0 + B_\emptyset)$ is a K-quasi-periodic decomposition of Y. If $|B_\emptyset| \ge 2$, then Item 2 yields $H \le K$ with $y_0 \in a_0 + B_0$ and $(a_0 + B_\emptyset) \setminus \{y_0\}$ H-periodic. Letting $b_0 = y_0 - a_0 \in B_0$, Item 3 follows. Therefore instead assume $|B_\emptyset| = 1$, say $B_\emptyset = \{b_0\}$. Moreover, in view of [35, Proposition 2.2], we can assume $a_0 + b_0 \in A + B$ is the only unique expression element.

In this case $\Big((A+B) \setminus \{a_0 + b_0\}\Big) \cup \{a_0 + b_0\}$ is a reduced K-quasi-periodic decomposition, in which case Item 1 implies $a_0 + b_0 = y_0$ and $(A+B) \setminus \{a_0 + b_0\} = Y \setminus \{y_0\}$. Since $(y_0 + H) \cap (A+B) = (y_0 + H) \cap Y = \{y_0\}$ with $y_0 = a_0 + b_0$, we must have $(a_0 + H) \cap A = \{a_0\}$ and $(b_0 + H) \cap B = \{b_0\}$. If $|A_0| = |B_0| = 1$, then Item 3 follows trivially. Therefore we can instead w.l.o.g. assume $|A_0| \ge 2$. Since $a_0 + b_0 = y_0 \in A + B = Y$ is the only unique expression, we have $(A \setminus \{a_0\}) + B = Y \setminus \{y_0\}$

with $\mathsf{H}((A\setminus\{a_0\})+B)=\mathsf{H}(Y\setminus\{y_0\})=H$. Kneser's Theorem now implies

$$|A|+|B|-2=|(A\setminus\{a_0\})+B|\geq|(A\setminus\{a_0\})+H|+|B+H|-|H|\geq|A\setminus\{a_0\}|+|B|-1, \tag{13}$$

with the latter inequality in view of $(b_0+H)\cap B=\{b_0\}$. Thus we must have equality in (13). In particular, equality holding in the second inequality in (13) forces $(A\setminus\{a_0\})+H=A\setminus\{a_0\}$ and $(B\setminus\{b_0\})+H=B\setminus\{b_0\}$, i.e., $A\setminus\{a_0\}$ and $B\setminus\{b_0\}$ are H-periodic, completing Item 3. □

Lemma 2.6. *Let G be an abelian group and let A, $B\subseteq G$ be finite, nonempty subsets with $H=\mathsf{H}(A+B)=\mathsf{H}(B)$. Then $|(A\cup\{x\})+B|\geq|(A\cup\{x\})+H|+|B+H|-|H|$ for any $x\in G$.*

Proof. By hypothesis, $\phi_H(A)+\phi_H(B)$ is aperiodic. Thus $(\phi_H(A)\cup\{\phi_H(x)\})+\phi_H(B)$ is either still aperiodic, in which case Kneser's Theorem implies $|\phi_H(A\cup\{x\})+\phi_H(B)|\geq|\phi_H(A\cup\{x\})|+|\phi_H(B)|-1$, or else it is periodic, and thus strictly contains the aperiodic subset $\phi_H(A)+\phi_H(B)$. In such case, applying Kneser' Theorem to $\phi_H(A)+\phi_H(B)$ yields $|\phi_H(A\cup\{x\})+\phi_H(B)|\geq|\phi_H(A)|+|\phi_H(B)|\geq|\phi_H(A\cup\{x\})|+|\phi_H(B)|-1$. In either case, since $(A\cup\{x\})+B$ is H-periodic in view of $H=\mathsf{H}(B)$, the desired conclusion follows by multiplying the inequality by $|H|$.

□

Lemma 2.7. *Let G be an abelian group, let $n\geq 1$, let $X\subseteq G$ be a finite, nonempty subset, let $S\in\mathcal{F}(G)$ be a sequence, and let $\mathcal{A}=A_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}A_n$ be a setpartition with $\mathsf{S}(\mathcal{A})=S$ and $|X+\sum_{i=1}^{n}A_i|\geq\min\{|S|-n+|X|,\ |X+\Sigma_n(S)|\}$. Then one of the following holds.*

1. *$n=2$, $X\setminus\{\beta\}$ and $A_1\cap A_2$ are K-periodic, $(A_1\cup A_2)\setminus(A_1\cap A_2)$ is a K-coset, $X+\Sigma_n(S)=X+A_1+A_2$ is aperiodic and $|X+\Sigma_n(S)|=|S|-n+|X|$, for some $\beta\in X$ and $K\leq G$ with $K\cong(\mathbb{Z}/2\mathbb{Z})^2$.*
2. *There exists an* equitable *setpartition $\mathcal{B}=B_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}B_n$ with $\mathsf{S}(\mathcal{B})=S$ and $|X+\sum_{i=1}^{n}B_i|\geq\min\{|S|-n+|X|,\ |X+\Sigma_n(S)|\}$. Moreover, if $|X+\Sigma_n(S)|\leq|S|-n+|X|$ and $|A_i\setminus Z|\leq 1$ for all i, where $H=\mathsf{H}(X+\sum_{i=1}^{n}A_i)$ and $Z=\bigcap_{i=1}^{n}(A_i+H)$, then $Z=\bigcap_{i=1}^{n}(B_i+H)$, $X+\sum_{i=1}^{n}B_i=X+\sum_{i=1}^{n}A_i=X+\Sigma_n(S)$ and $|B_i\setminus Z|\leq 1$ for all i.*

Proof. If $|S|=\sum_{i=1}^{n}|A_i|\leq n+1$ or $n=1$, then $\mathcal{A}$ is trivially equitable, and Item 2 follows taking $\mathcal{B}$ to be $\mathcal{A}$. Therefore we may assume $|S|\geq n+2$ and $n\geq 2$. In particular, G is nontrivial. Let $\mathcal{A}=A_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}A_n$ be an arbitrary setpartition with $\mathsf{S}(\mathcal{A})=S$ and

$$|X+\sum_{i=1}^{n}A_i|\geq\min\{|X|+\sum_{i=1}^{n}|A_i|-n,|X+\Sigma_n(S)|\}. \tag{14}$$

Note $\mathcal{A}$ exists by hypothesis. If, for the original setpartition $\mathcal{A}$, we have $|X+\Sigma_n(S)|\leq|S|-n+|X|$ and $|A_i\setminus Z|\leq 1$ for all i, where $H=\mathsf{H}(X+\sum_{i=1}^{n}A_i)$ and $Z=\bigcap_{i=1}^{n}(A_i+H)$, then fix the set

$Z \subseteq G$ and only consider setpartitions $\mathcal{A}$ also satisfying

$$Z \subseteq \bigcap_{i=1}^{n}(A_i+H) \quad \text{and} \quad |A_i \setminus Z| \leq 1 \quad \text{for all } i, \tag{15}$$

for the fixed set Z. Otherwise, simply let $Z = H = G$. In either case, let $e = \sum\limits_{i=1}^{n}|A_i \setminus Z|$.

Note $|X + \Sigma_n(S)| \leq |S| - n + |X| = |X| + \sum\limits_{i=1}^{n}|A_i| - n$ and $|A_i \setminus Z| \leq 1$ for all i combined with (14) imply $X + \sum\limits_{i=1}^{n} A_i = X + \Sigma_n(S)$, combined with $\sum\limits_{i=1}^{n}|A_i| = |S| \geq n+2$ imply Z is nonempty, and combined with the definition of e imply $e \leq n$. Moreover, if $e = n$, then the n terms from the sets $A_i \setminus Z$ for $i = 1, \ldots, n$ cannot all be equal modulo H, else they would be included in the set Z by definition. What this means is that an arbitrary setpartition $\mathcal{A}$ satisfying (14) and (15) must have $X + \sum\limits_{i=1}^{n} A_i = X + \Sigma_n(S)$ and $Z = \bigcap_{i=1}^{n}(A_i + H)$, so that the quantities $H = \mathsf{H}(X + \sum\limits_{i=1}^{n} A_i) = \mathsf{H}(X + \Sigma_n(S))$ and $Z = \bigcap_{i=1}^{n}(A_i + H)$ remain invariant as we range over all setpartitions satisfying (14) and (15). This is also trivially the case when $Z = H = G$. Now choose a setpartition $\mathcal{A}$ with $\mathsf{S}(\mathcal{A}) = S$ satisfying (14) and (15) that is as equitable as possible, meaning one such that

$$\sum_{i=1}^{n}|A_i|^2 \quad \text{is minimal.} \tag{16}$$

Assume by contradiction that $\mathcal{A}$ is not equitable, so

$$m := \min_{i \in [1,n]}\{|A_i|\} \leq \max_{i \in [1,n]}\{|A_i|\} - 2.$$

Note this ensures that H is nontrivial, lest $|A_i| \in \{|Z|, |Z|+1\}$ for all i. Let $I_m \subseteq [1,n]$ be the subset of indices $i \in [1,n]$ with $|A_i| = m$, let $I_{m+1} \subseteq [1,n]$ be the subset of indices $i \in [1,n]$ with $|A_i| = m+1$, let $I_{m+2} \subseteq [1,n]$ be the subset of indices $i \in [1,n]$ with $|A_i| \geq m+2$, let $I_Z \subseteq [1,n]$ be all indices $i \in [1,n]$ with $A_i \subseteq Z$, let $I_e \subseteq [1,n]$ be all indices $i \in [1,n]$ with $A_i \setminus Z$ nonempty, and let

$$J_Z = (I_m \cup I_{m+1}) \cap I_Z \quad \text{and} \quad J_e = (I_m \cup I_{m+1}) \cap I_e.$$

Since $\mathcal{A}$ is not equitable, I_{m+2} and I_m are nonempty. Consider $k \in I_{m+2}$ and $s \in I_m$. Since $k \in I_{m+2}$ and $|A_k \setminus Z| \leq 1$, we have $|A_k \cap Z| \geq |A_k| - 1 \geq m+1$. Let $\alpha_1, \ldots, \alpha_r \in A_k \cap Z$ be those elements contained in Z with $(\alpha_i + H) \cap A_k = \{\alpha_i\}$, and let $Z_1 = \{\alpha_1, \ldots, \alpha_r\} + H \subseteq Z$. Then $|A_k \cap (Z \setminus Z_1)| = |A_k \cap Z| - r \geq |A_k| - 1 - r \geq m + 1 - r$. Since $s \in I_m$ and $Z \subseteq A_s + H$, it follows that $|(Z \setminus Z_1) \cap A_s| \leq |A_s \cap Z| - r \leq |A_s| - r = m - r$, in which case the pigeonhole principle guarantees there is some $y \in (A_k \setminus A_s) \cap Z$ with $|(y+H) \cap A_k| \geq 2$. Consequently, the indices k and s and element y in the hypothesis of the following claim always exist. Moreover, if $k\in I_{m+2}\cap I_Z$, then we get the improved estimate $|A_k\cap Z|=|A_k|\geq m+2$, in which case the above argument yields at least *two* elements $y,\, y'\in A_k$ satisfying the hypotheses of Claim A.

Claim A. $A_s\nsubseteq A_k$ and $X+\sum_{\substack{i=1\\ i\neq k}}^{n}A_i+(A_k\setminus\{y\})\neq X+\sum_{i=1}^{n}A_i$ for any $k\in I_{m+2}$, $s\in I_m$ and $y\in (A_k\setminus A_s)\cap Z$ with $|(y+H)\cap A_k|\geq 2$.

Proof. If $X+\sum_{\substack{i=1\\ i\neq k}}^{n}A_i+(A_k\setminus\{y\})=X+\sum_{i=1}^{n}A_i$, then we can remove y from A_k and place it in A_s to yield a new setpartition $\mathcal{B}=B_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}B_n$, where $B_k=A_k\setminus\{y\}$, $B_s=A_s\cup\{y\}$, and $B_i=A_i$ for all $i\neq k,s$, such that $\mathsf{S}(\mathcal{B})=S$, $X+\sum_{i=1}^{n}A_i\subseteq X+\sum_{i=1}^{n}B_i$, $Z\subseteq\bigcap_{i=1}^{n}(B_i+H)$ (since $|(y+H)\cap A_k|\geq 2$), and $|B_i\setminus Z|=|A_i\setminus Z|\leq 1$ for all i (since $y\in Z$). Thus $\mathcal{B}$ satisfies (14) and (15). However, since $|A_k|\geq |A_s|+2$, it follows that $|A_k|^2+|A_s|^2>|B_k|^2+|B_s|^2$, so that $\mathcal{B}$ contradicts the minimality of (16) for $\mathcal{A}$. Therefore we instead conclude that $X+\sum_{\substack{i=1\\ i\neq k}}^{n}A_i+(A_k\setminus\{y\})\neq X+\sum_{i=1}^{n}A_i$.

If $A_s\subseteq A_k$, then let $y\in(A_k\setminus A_s)\cap Z$ be any element with $|(y+H)\cap A_k|\geq 2$, which exists as argued above Claim A. Then $A_s\subseteq A_k\setminus\{y\}$, whence $A_s+A_k\subseteq(A_s\cup\{y\})+(A_k\setminus\{y\})$, and removing y from A_k and placing it in A_s yields a new setpartition $\mathcal{B}$ that contradicts the minimality of (16) for $\mathcal{A}$ as before. Therefore $A_s\nsubseteq A_k$, completing the claim. □

Let $L=\mathsf{H}(X+\sum_{i\in I_m\cup I_{m+1}}A_i)$ and re-index the A_i such that $J_Z=(I_m\cup I_{m+1})\cap I_Z=[1,n_Z]$, $J_e=(I_m\cup I_{m+1})\cap I_e=[n_Z+1,n_e]$ and $I_{m+2}=[n_e+1,n]$.

Claim B. $|A_k+L|\geq |A_k|+3(|L|-1)$ for any $k\in I_{m+2}$.

Proof. Let $k\in I_{m+2}$ and $s\in I_m$. Since $X+\sum_{\substack{i=1\\ i\neq k}}^{n}A_i$ is L-periodic (as $k\notin I_m\cup I_{m+1}$), it follows from Claim A that any element $y\in A_k\cap Z$ satisfying the hypotheses of Claim A must be the unique element from its L-coset in A_k. Since $L\leq H$, the same is true of any element $y\in A_k$ which is the unique element from its H-coset in A_k. Since there is always at least one element satisfying the hypotheses of Claim A, at least two when $k\in I_Z$, and also an element from $A_k\setminus Z$ which is the unique element from its H-coset in A_k when $k\in I_e$, the claim follows if there is any $y\in A_k\cap Z$ that is the unique element from its H-coset in A_k, so we instead assume $|(y+H)\cap A_k|\geq 2$ for all $y\in A_k\cap Z$. Thus

$$|(A_k\cap Z)\setminus A_s|\geq |A_k\cap Z|-|A_s|+1=|A_k\cap Z|-m+1,$$

is the number of elements $y\in A_k\cap Z$ satisfying the hypothesis of Claim A, with the inequality following since Claim A ensures that $A_s\nsubseteq A_k$. If $k\in I_Z$, we obtain at least $|(A_k\cap Z)\setminus A_s|\geq |A_k\cap Z|-m+1=|A_k|-m+1\geq 3$ elements satisfying the hypotheses of Claim A. If $j\in I_e$, we obtain at least $|(A_k\cap Z)\setminus A_s|\geq |A_k\cap Z|-m+1\geq |A_k|-m\geq 2$ elements satisfying the hypotheses of Claim A, as well as the element from $A_k \setminus Z$, which is the unique element from its H-coset in A_k. In either case, the claim follows. $\square$

Claim C. Either $|X + \sum_{i\in J_Z} A_i| \leq |X| + \sum_{i\in J_Z} |A_i| - |J_Z| - \max\{|L|-1,\, 1\}$, or else L is trivial and $|X + \sum_{i=1}^{j} A_i| = |X + \sum_{i=1}^{j-1} A_i| + |A_j| - 1$ for all $j \in [1,n]$.

Proof. Suppose

$$|X + \sum_{i\in J_Z} A_i| \geq |X| + \sum_{i\in J_Z} |A_i| - |J_Z| - \max\{|L|-1,\, 1\} + 1. \tag{17}$$

Note $J_Z \cup J_e = I_m \cup I_{m+1}$ and $L = \mathsf{H}(X + \sum_{i\in J_Z} A_i + \sum_{i\in J_e} A_i)$. Kneser' Theorem implies

$$|X + \sum_{i\in I_m\cup I_{m+1}} A_i| \geq |X + \sum_{i\in J_Z} A_i| + \sum_{i\in J_e} |A_i + L| - |J_e||L|. \tag{18}$$

By definition, each A_i with $i \in J_e \subseteq I_e$ has some element $z \in A_i \setminus Z$ which is the unique element from its H-coset in A_i, meaning $(z+H)\cap A_i = \{z\}$. Since $L \leq H$, it is also the unique element from its L-coset in A_i, ensuring that $|A_i + L| \geq |A_i| + |L| - 1$ for $i \in J_e$. Combining this observation with (17) and (18), we conclude that

$$|X + \sum_{i\in I_m\cup I_{m+1}} A_i| \geq |X| + \sum_{i\in I_m\cup I_{m+1}} |A_i| - |I_m \cup I_{m+1}| - \max\{|L|-1,\, 1\} + 1. \tag{19}$$

Let $k \in I_{m+2} = [n_e+1, n]$. Since $X + \sum_{i=1}^{n_e} A_i$ is L-periodic, it follows that $X + \sum_{i=1}^{k-1} A_i$ is also L-periodic. Consequently, in view of Claim A, it follows that there is a unique expression element in the sumset $\phi_L\Big(X + \sum_{i=1}^{k-1} A_i\Big) + \phi_L(A_k)$. Thus Theorem D implies that $|\phi_L\Big(X + \sum_{i=1}^{k-1} A_i\Big) + \phi_L(A_k)| \geq |\phi_L\Big(X + \sum_{i=1}^{k-1} A_i\Big)| + |\phi_L(A_k)| - 1$ for $k \in I_{m+2}$. Lemma 2.1 now yields

$$|X + \sum_{i=1}^{n} A_i| \geq |X + \sum_{i\in I_m\cup I_{m+1}} A_i| + \sum_{i\in I_{m+2}} |A_i + L| - |I_{m+2}| - (|L|-1)|I_{m+2}|. \tag{20}$$

By Claim B, we have $|A_k + L| \geq |A_k| + 3(|L|-1)$ for all $k \in I_{m+2}$. Combining this observation with (19) and (20), we deduce that

$$|X + \sum_{i=1}^{n} A_i| \geq |X| + \sum_{i=1}^{n} |A_i| - n + 2|I_{m+2}|(|L|-1) - \max\{|L|-1, 1\} + 1. \tag{21}$$

Suppose $|X + \sum_{i=1}^{n} A_i| \leq |X| + \sum_{i=1}^{n} |A_i| - n$. Then, since $|I_{m+2}| \geq 1$, we must have equality in (21) with L trivial, and equality must hold in all estimates used to derive (21). Since L is trivial, Kneser's Theorem implies $|X + \sum_{i=1}^{j} A_i| \geq |X| + \sum_{i=1}^{j} |A_i| - j$ for all $j \in I_m \cup I_{m+1} = [1, n_e]$.

But now Lemma 2.1 ensures that equality must hold in all these estimates, as otherwise (19) holds strictly, implying (21) also holds strictly. We also have equality holding in the modulo L estimates used to derive (20), meaning $|X+\sum_{i=1}^{k}A_i| = |X|+\sum_{i=1}^{k}|A_i|-k$ for all $k\geq n_e+1$, for otherwise (20) holds strictly, and thus (21) as well. The claim now follows. So it remains to consider the case when

$$|X+\sum_{i=1}^{n}A_i| > |X|+\sum_{i=1}^{n}|A_i|-n = |S|-n+|X|,$$

in which case $Z=H=G$.

In this case, since $n\in I_{m+2}$ by our choice of indexing, Claim A ensures that every element $y\in A_n\setminus A_s$ is part of a unique expression element in $\left(X+\sum_{i=1}^{n-1}A_i\right)+A_n$, where $s\in I_m$. If some $y\in A_n\setminus A_s$ is part of exactly one unique expression element, then $|X+\sum_{i=1}^{n-1}A_i+(A_n\setminus\{y\})| = |X+\sum_{i=1}^{n}A_i|-1\geq |X|+\sum_{i=1}^{n}|A_i|-n$. Thus removing y from A_n and placing it in A_s yields a new setpartition contradicting the minimality of (16) for $\mathcal{A}$. Therefore we can assume each $y\in A_n\setminus A_s$ is part of at least two unique expression elements. Since $A_s\nsubseteq A_n$ by Claim A, we have $|A_n\setminus A_s|\geq 3$, so there are at least three such elements in A_n, and thus at least two elements in $A_n\setminus\{y\}$ that are each part of at least two unique expression elements in the sumset $\left(X+\sum_{i=1}^{n-1}A_i\right)+(A_n\setminus\{y\})$, for any fixed $y\in A_n\setminus A_s$. In consequence, the Kemperman Structure Theorem [37, Theorem 9.1] [35, Proposition 2.2] implies

$$|\left(X+\sum_{i=1}^{n-1}A_i\right)+(A_n\setminus\{y\})|\geq |X+\sum_{i=1}^{n-1}A_i|+|A_n\setminus\{y\}|. \tag{22}$$

If L is trivial or $|I_{m+2}|\geq 2$, then repeating the arguments used to derive (20) and (21) for $A_1\cdot\ldots\cdot A_{n-1}$ rather than $A_1\cdot\ldots\cdot A_n$ yields $|X+\sum_{i=1}^{n-1}A_i|\geq |X|+\sum_{i=1}^{n-1}|A_i|-(n-1)$. Combined with (22), it follows that $|X+\sum_{i=1}^{n-1}A_i+(A_n\setminus\{y\})|\geq |X|+\sum_{i=1}^{n}|A_i|-n$, and now removing y from A_n and placing it in A_s yields a new setpartition contradicting the minimality of (16) for $\mathcal{A}$. Therefore we now assume L is nontrivial and $|I_{m+2}|=1$, meaning $I_m\cup I_{m+1}=[1,n-1]$. In this case, since $H=G$ ensures $n\in I_Z\cap I_{m+2}$ so that there is at least one element from $A_n\setminus\{y\}$ satisfying the hypothesis of Claim A, we can repeat the arguments used to derive (20) for $A_1\cdot\ldots\cdot A_{n-1}\cdot(A_n\setminus\{y\})$ rather than $A_1\cdot\ldots\cdot A_n$ to find

$$|X+\sum_{i=1}^{n-1}A_i+(A_n\setminus\{y\})|\geq |X+\sum_{i=1}^{n-1}A_i|+|(A_n\setminus\{y\})+L|-|L|\geq |X+\sum_{i\in I_m\cup I_{m+1}}A_i|+|A_n|+|L|-3, \tag{23}$$

with the latter inequality as Claim B ensures $|(A_n\setminus\{y\})+L|-|A_n\setminus\{y\}|\geq 2(|L|-1)$. Combining (23) with (19) and using that L is nontrivial, we obtain $|X+\sum_{i=1}^{n-1}A_i+(A_n\setminus\{y\})|\geq |X|+\sum_{i=1}^{n}|A_i|-n$. Once again, removing y from A_n and placing it in A_s yields a new setpartition contradicting the minimality of (16) for $\mathcal{A}$, completing Claim C. □

We now split the proof into two cases depending on which outcome holds in Claim C.

CASE 1. $|X+\sum_{i\in J_Z}A_i|\leq |X|+\sum_{i\in J_Z}|A_i|-|J_Z|-\max\{|L|-1,\,1\}$ holds for every setpartition $\mathcal{A}=A_1\cdot\ldots\cdot A_n$ with $\mathsf{S}(\mathcal{A})=S$ satisfying (14), (15) and (16).

For $j\in[0,n_Z]$, let $K_j=\mathsf{H}(X+\sum_{i=1}^{j}A_i)\leq L$. In view of the case hypothesis, let $j\in J_Z=[1,n_Z]$ be the minimal index such that $|X+\sum_{i=1}^{j}A_i|\leq |X|+\sum_{i=1}^{j}|A_i|-j-\max\{|K_j|-1,\,1\}$.

Claim D. $K_j=\mathsf{H}(X+\sum_{i=1}^{j}A_i)=\mathsf{H}(X+\sum_{i=1}^{j-1}A_i)$ and $|X+\sum_{i=1}^{j}A_i|<|X+\sum_{i=1}^{j-1}A_i|+|A_j|-1$.

Proof. If $j=1$, then $|X+A_1|\leq |X|+|A_1|-2$, so that Kneser's Theorem implies K_1 is nontrivial with $|X|+|A_1|-|K_1|\geq |X+A_1|\geq |X+K_1|+|A_1|-|K_1|$ (both upper bounds for $|X+A_1|$ follow from the definition of j). Thus $X+K_1=X$, ensuring that $K_0=K_1$, and the claim follows. Therefore we now assume $j\in[2,n_Z]$. Hence the minimality of j ensures

$$|X+\sum_{i=1}^{j-1}A_i|\geq |X|+\sum_{i=1}^{j-1}|A_i|-(j-1)-\max\{|K_{j-1}|-1,\,1\}+1. \tag{24}$$

Applying Kneser's Theorem again, we find

$$|X+\sum_{i=1}^{j}A_i|\geq |X+\sum_{i=1}^{j-1}A_i+K_j|+|A_j|-|K_j|. \tag{25}$$

If $K_{j-1}\neq K_j$, then K_j is nontrivial and $|X+\sum_{i=1}^{j-1}A_i+K_j|\geq |X+\sum_{i=1}^{j-1}A_i|+|K_{j-1}|$, which combined with (24) and (25) yields $|X+\sum_{i=1}^{j}A_i|\geq |X|+\sum_{i=1}^{j}|A_i|-j+1-(|K_j|-1)=|X|+\sum_{i=1}^{j}|A_i|-j+1-\max\{|K_j|-1,\,1\}$, contrary to the definition of j. Therefore $K_{j-1}=K_j$. If $|X+\sum_{i=1}^{j}A_i|\geq |X+\sum_{i=1}^{j-1}A_i|+|A_j|-1$, then (24) yields $|X+\sum_{i=1}^{j}A_i|\geq |X|+\sum_{i=1}^{j}|A_i|-j+1-\max\{|K_{j-1}|-1,\,1\}=|X|+\sum_{i=1}^{j}|A_i|-j+1-\max\{|K_j|-1,\,1\}$, again contrary to the definition of j. Therefore $|X+\sum_{i=1}^{j}A_i|<|X+\sum_{i=1}^{j-1}A_i|+|A_j|-1$, and the claim follows. □

All the above is valid for any $\mathcal{A}$ with $\mathsf{S}(\mathcal{A}) = S$ which satisfies (14), (15) and (16). We now impose additional extremal conditions on $\mathcal{A}$:

(a) $|J_Z|$ is minimal (subject to satisfying (14), (15) and (16)), say $|J_Z| = n_Z$, with the A_i indexed so that $J_Z = [1, n_Z]$.
(b) j is maximal (subject to satisfying (14), (15), (16) and (a)).
(c) $\sum\limits_{i=1}^{j} |\phi_K(A_i)|$ is maximal, where $K = \mathsf{H}(X + \sum\limits_{i=1}^{j-1} A_i)$ (subject to satisfying (14), (15), (16), (a) and (b)).

Let $k \in I_{m+2}$ be fixed and set $K := K_j \leq L \leq H$, so

$$K = \mathsf{H}(X + \sum_{i=1}^{j-1} A_i) = \mathsf{H}(X + \sum_{i=1}^{j} A_i)$$

by Claim D. Since Claim D ensures $|(X + \sum\limits_{i=1}^{j-1} A_i) + A_j| < |X + \sum\limits_{i=1}^{j-1} A_i| + |A_j| - 1$, it follows from Kneser's Theorem applied to $(X + \sum\limits_{i=1}^{j-1} A_i) + A_j$ that K is nontrivial and

$$|A_j + K| - |A_j| \leq |K| - 2. \tag{26}$$

In particular,

$$|(x + K) \cap A_j| \geq 2 \quad \text{for all } x \in A_j. \tag{27}$$

Since $j \in J_Z \subseteq I_Z$, we have $A_j \subseteq Z$. Let $Z_0 = (A_j + K) \setminus (A_k + K) \subseteq Z$. Let $X_0 \subseteq Z_0 \cap A_j$ be a subset consisting of one element from A_j for every K-coset contained in Z_0. Thus

$$|Z_0| = |K|\,|X_0|$$

with Z_0 the union of all K-cosets that intersect A_j but not A_k.

Claim E. There is a subset $Y \subseteq (A_k \cap Z) \setminus (A_j + K)$ with $|Y| \geq \max\{1, |X_0|\}$ and $(A_k \setminus Y) + H = A_k + H$.

Proof. Let us first show there is some $y \in (A_k \cap Z) \setminus (A_j + K)$ with $|(y + H) \cap A_k| \geq 2$. Let $y_0 \in A_k \cap Z$ be an element satisfying the hypotheses of Claim A. Then $|(y_0 + H) \cap A_k| \geq 2$. Moreover, since $X + \sum\limits_{i \in I_m \cup I_{m+1}} A_i$ is K-periodic with $k \in I_{m+2}$, the conclusion of Claim A ensures that $(y_0 + K) \cap A_k = \{y_0\}$. If $y_0 \notin A_j + K$, then taking $y = y_0$ yields the desired element y. Therefore we may assume $y_0 \in A_j + K$. Then $Z_0 \cup (y_0 + K) \subseteq A_j + K$ with $\Big(Z_0 \cup (y_0 + K)\Big) \cap A_k = \{y_0\}$. Thus $|A_k \cap (A_j + K)| \leq |A_j + K| - |Z_0| - |K| + 1 \leq |A_j| - |Z_0| - 1$, with the second inequality in view of (26). It follows that

$$|(A_k \cap Z) \setminus (A_j + K)| = |A_k \cap Z| - |A_k \cap (A_j + K)| \geq (|A_k| - 1) - (|A_j| - |Z_0| - 1) = |A_k| - |A_j| + |Z_0|. \tag{28}$$

In particular, since $|A_k|-|A_j|\geq (m+2)-(m+1)=1$ (as $k\in I_{m+2}$ and $j\in I_m\cup I_{m+1}$), we conclude that $(A_k\cap Z)\setminus(A_j+K)$ is nonempty. If there is some $y\in(A_k\cap Z)\setminus(A_j+K)$ with $|(y+H)\cap A_k|\geq 2$, then the desired element y is found. Otherwise, we conclude that each $y\in(A_k\cap Z)\setminus(A_j+K)$ is the unique element from its H-coset in A_k. However, since $y\in Z\subseteq A_j+H$, it follows that $(y+H)\cap A_j$ is also nonempty, say with $y'\in(y+H)\cap A_j$. Since $y\notin A_j+K$, we have $y\notin y'+K$, ensuring that $y'+K\neq y+K$. Thus, as $(y+H)\cap A_k=\{y\}$, we conclude that $(y'+K)\cap A_k$ is empty, meaning $y'+K\subseteq Z_0$. As this is true for each $y\in(A_k\cap Z)\setminus(A_j+K)$, with the corresponding sets $y'+K$ each lying in the distinct cosets $y+H$ for $y\in(A_k\cap Z)\setminus(A_j+K)$ (as each such y is the unique element from its H-coset in A_k), it follows that

$$|Z_0|\geq|(A_k\cap Z)\setminus(A_j+K)|\,|K|\geq|(A_k\cap Z)\setminus(A_j+K)|.$$

However, applying this estimate in (28) yields the contradiction $m+2\leq|A_k|\leq|A_j|\leq m+1$. Thus the existence of the desired element $y\in(A_k\cap Z)\setminus(A_j+K)$ with $|(y+H)\cap A_k|\geq 2$ is established, and we assume $|X_0|\geq 2$ as the claim is now complete taking $Y=\{y\}$ when $|X_0|\leq 1$.

By definition, $Z_0\cap A_k=\emptyset$ and $Z_0\subseteq A_j+K$. Thus $|A_k\cap(A_j+K)|\leq|A_j+K|-|Z_0|\leq |A_j|-|Z_0|+|K|-2=|A_j|-|K|(|X_0|-1)-2$, with the second inequality in view of (26). It follows that

$$|(A_k\cap Z)\setminus(A_j+K)|=|A_k\cap Z|-|A_k\cap(A_j+K)|\geq|A_k|-|A_j|+|K|(|X_0|-1)+1. \tag{29}$$

Let $Z_0'=\Big((A_k\cap Z)\setminus(A_j+K)+H\Big)\cap(Z_0+H)$ and partition $\Big((A_k\cap Z)\setminus(A_j+K)+H\Big)=Z_0'\cup Z_1$. Since each H-coset in Z_0' contains a K-coset from Z_0, we have

$$|X_0|\geq|Z_0'|/|H|. \tag{30}$$

Let $Y\subseteq(A_k\cap Z)\setminus(A_j+K)$ be obtained by taking the set $(A_k\cap Z)\setminus(A_j+K)$ and removing one element from $(A_k\cap Z)\setminus(A_j+K)$ from each of the $|Z_0'|/|H|$ H-cosets contained in Z_0'. Then

$$|Y|=|(A_k\cap Z)\setminus(A_j+K)|-|Z_0'|/|H|\geq|(A_k\cap Z)\setminus(A_j+K)|-|X_0|, \tag{31}$$

with the inequality in view of (30), and $(\alpha+H)\cap(A_k\setminus Y)$ is nonempty for every $\alpha+H\subseteq Z_0'$ (as one element for each of these H-cosets was left out of Y, and thus remains in $A_k\setminus Y$). For each $\alpha+H\subseteq Z_1\subseteq Z$, we have $\alpha+H\subseteq Z\subseteq A_j+H$, and thus there is some $\alpha'\in\alpha+H$ with $(\alpha'+K)\cap A_j$ nonempty. Since $\alpha+H\nsubseteq Z_0'$, it follows by definition of Z_0' and Z_0 that $(\alpha'+K)\cap A_k$ is nonempty, and necessarily disjoint from $(A_k\cap Z)\setminus(A_j+K)$, and thus also from $Y\subseteq(A_k\cap Z)\setminus(A_j+K)$. In consequence, we have $(A_k\setminus Y)+H=A_k+H$. If $|Y|\geq|X_0|$, the claim is complete, so we instead assume $|Y|\leq|X_0|-1$, in which case (31) implies $|(A_k\cap Z)\setminus(A_j+K)|\leq 2|X_0|-1$. However, using this estimate in (29) along with $|A_k|-|A_j|\geq(m+2)-(m+1)=1$ yields $(|K|-2)(|X_0|-1)+1\leq 0$, which is not possible since K is nontrivial (as noted above (26)) and $|X_0|\geq 1$, completing the claim. □

In view of Claim E, there is a nonempty subset $Y \subseteq (A_k \cap Z) \setminus (A_j + K)$ with $(A_k \setminus Y) + H = A_k + H$ and $|Y| = \max\{1, |X_0|\}$. Define a new setpartition $\mathcal{B} = B_1 \cdot \ldots \cdot B_n$ by setting $B_j = (A_j \setminus X_0) \cup Y$, $B_k = (A_k \setminus Y) \cup X_0$ and $B_i = A_i$ for all $i \neq k, j$. Since $K = \mathsf{H}(X + \sum_{i=1}^{j-1} A_i)$, (27) ensures that $\phi_K(A_j) = \phi_K(A_j \setminus X_0)$ and $X + \sum_{i=1}^{j} A_i = X + \sum_{i=1}^{j-1} A_i + (A_j \setminus X_0)$. As a result,

$$X + \sum_{i=1}^{n} A_i \subseteq X + \sum_{\substack{i=1 \\ i \neq j,k}}^{n} A_i + (A_j \setminus X_0) + (A_k \cup X_0). \tag{32}$$

By definition of Y, we have $\phi_K(Y)$ disjoint from $\phi_K(A_j)$, and thus also from $\phi_K(X_0)$, while the definition of X_0 ensures that $\phi_K(X_0)$ is disjoint from $\phi_K(A_k)$ with $\phi_K(A_j \setminus X_0) = \phi_K(A_j) \subseteq \phi_K(A_k \cup X_0) \setminus \phi_K(Y)$. It follows that

$$\begin{aligned} \phi_K(A_j \setminus X_0) + \phi_K(A_k \cup X_0) &\subseteq \Big(\phi_K(A_j \setminus X_0) \cup \phi_K(Y)\Big) + \Big(\phi_K(A_k \cup X_0) \setminus \phi_K(Y)\Big) \\ &\subseteq \phi_K(B_j) + \phi_K(B_k). \end{aligned}$$

As a result, since $X + \sum_{i=1}^{j-1} A_i = X + \sum_{i=1}^{j-1} B_i$ is K-periodic, we conclude from (32) that

$$X + \sum_{i=1}^{n} A_i \subseteq X + \sum_{\substack{i=1 \\ i \neq j,k}}^{n} A_i + (A_j \setminus X_0) + (A_k \cup X_0) \subseteq X + \sum_{i=1}^{n} B_i,$$

so (14) holds for $\mathcal{B}$. Since $X_0 \subseteq A_j \subseteq Z$ (as $j \in J_Z$), Claim E ensures that $Z \subseteq A_k + H = B_k + H$ and $|B_k \setminus Z| = |A_k \setminus Z| \leq 1$. Since $\phi_K(A_j) \subseteq \phi_K(B_j)$ with $Y \subseteq Z$ by definition, we have $Z \subseteq B_j + H = A_j + H$ and $|B_j \setminus Z| = |A_j \setminus Z| = 0$. Consequently, $Z \subseteq \bigcap_{i=1}^{n}(B_i + H)$ and $|B_i \setminus Z| \leq 1$ for all i, in which case $\mathcal{B}$ satisfies (15). We have $|B_j| = |A_j| - |X_0| + |Y|$, $|B_k| = |A_j| - |Y| + |X_0|$, and $|B_i| = |A_i|$ for $i \neq j, k$. Let I'_m, I'_{m+1}, I'_{m+2}, I'_e, I'_Z and J'_Z be the associated quantities I_m, I_{m+1}, I_{m+2}, I_e, I_Z and J_Z for $\mathcal{B}$ rather than $\mathcal{A}$.

Suppose $|X_0| = 0$. Then $|Y| = 1$, $|B_j| = |A_j| + 1$ and $|B_k| = |A_k| - 1$. If $|A_k| \geq |A_j| + 2$, then $\mathcal{B}$ contradicts the minimality of (16) for $\mathcal{A}$. Otherwise, we have $|A_k| = |A_j| + 1 = m + 2$, $|B_j| = |A_k| = m + 2$ and $|B_k| = |A_j| = m + 1$ so that $\sum_{i=1}^{n} |B_i|^2 = \sum_{i=1}^{n} |A_i|^2$, meaning $\mathcal{B}$ satisfies the extremal condition (16). Now $I'_{m+1} = I_{m+1} \setminus \{j\} \cup \{k\}$, $I'_m = I_m$ and $I'_{m+2} = I_{m+2} \setminus \{k\} \cup \{j\}$. If $k \in I_e$, then $k \in I'_e$ (as $Y \subseteq Z$), in which case $J'_Z = J_Z \setminus \{j\}$, contradicting the minimality condition (a) for $\mathcal{A}$. On the other hand, if $k \in I_Z$, then $J'_Z = J_Z \setminus \{j\} \cup \{k\}$, so condition (a) holds for $\mathcal{B}$. Swapping the indices on B_k and B_j, so now $B_k = (A_j \setminus X_0) \cup Y$ and $B_j = (A_k \setminus Y) \cup X_0$, we obtain $J'_Z = J_Z$ and $I'_{m+1} \cup I'_m = I_{m+1} \cup I_m$. Since $A_i = B_i$ for $i < j$, the definition of j ensures $j' \geq j$, where j' is the associated quantity for $\mathcal{B}$ corresponding to the index j for $\mathcal{A}$, while the extremal condition given in (b) forces $j' \leq j$. Thus $j' = j$. However, since $k \in I_Z$, there are at least two elements y satisfying the hypotheses of Claim A for $\mathcal{A}$, which in view of the conclusion of Claim A and $K = \mathsf{H}(X + \sum_{i=1}^{j-1} A_i) = \mathsf{H}(X + \sum_{i=1}^{j'-1} B_i)$, means both these elements are the unique element from their K-coset in A_k. As at most one of them can be contained in the singleton set Y, we conclude that $B_j = A_k \setminus Y$ contains some $y \in B_j$ with $|(y+K) \cap B_j| = 1$. However, in such case, (27) could not hold for the index j in $\mathcal{B}$, contradicting that it must hold for $j' = j$ by the arguments above. So we instead conclude that $|X_0| \geq 1$,

Since $|X_0| \geq 1$, we have $|X_0| = |Y|$, $|B_j| = |A_j|$ and $|B_k| = |A_k|$, so (16) holds for $\mathcal{B}$ with $I'_m = I_m$, $I'_{m+1} = I_{m+1}$ and $I'_{m+2} = I_{m+2}$. Since $Y \subseteq Z$ and $A_j \subseteq Z$ (as $j \in J_Z$), we conclude that $J'_Z = J_Z$, meaning condition (a) holds for $\mathcal{B}$. As argued in the previous case, we must have $j' = j$, so that condition (b) holds. In particular, we must have $K = \mathsf{H}(X + \sum_{i=1}^{j-1} A_i) = \mathsf{H}(X + \sum_{i=1}^{j'-1} B_i)$. By definition of Y, each $y \in Y$ is disjoint from $A_j + K$. Thus, since $|Y| = |X_0| \geq 1$ and $\phi_K(A_j \setminus X_0) = \phi_K(A_j)$, we conclude that $|\phi_K(B_j)| > |\phi_K(A_j)|$, in which case the maximality condition (c) for $\mathcal{A}$ is contradicted by $\mathcal{B}$, completing CASE 1.

CASE 2. L is trivial and $|X + \sum_{i=1}^{j} A_i| = |X + \sum_{i=1}^{j-1} A_i| + |A_j| - 1$ for all $j \in [1, n]$, for some setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) = S$ satisfying (14), (15) and (16), where $J_Z = [1, n_Z]$, $J_e = [n_Z + 1, n_e]$ and $I_{m+2} = [n_e + 1, n]$.

Let $Y = X + \sum_{i=1}^{n-1} A_i$ and $V = X + \sum_{i=1}^{n} A_i = Y + A_n$. In view of the case hypothesis and Lemma 2.1, we have $|X + \sum_{i=1}^{j} A_i| = |X| + \sum_{i=1}^{j} |A_i| - j$ for all $j \in [1, n]$. In particular, $|X + \sum_{i \in J_Z} A_i| = |X| + \sum_{i \in J_Z} |A_i| - |J_Z|$ and

$$|V| = |X + \sum_{i=1}^{n} A_i| = |X| + \sum_{i=1}^{n} |A_i| - n = |S| - n + |X|. \tag{33}$$

The former equality combined with Claim C ensures that the hypotheses of CASE 2 hold for $\mathcal{A}$ under any re-indexing of the A_i with $i \in I_{m+2} = [n_e + 1, n]$, allowing us to freely assume an arbitrary set A_k with $k \in I_{m+2}$ occurs with $k = n$. We aim to either contradict the extremal condition (16) or show that Item 1 holds.

Claim F. If $H = \mathsf{H}(X + \sum_{i=1}^{n} A_i)$, then $A_k \subseteq Z$ with $|(y + H) \cap A_k| \geq 2$ for all $k \in I_{m+2}$ and $y \in A_k$.

Proof. Suppose $H = \mathsf{H}(X + \sum_{i=1}^{n} A_i)$. Note H is nontrivial as remarked after (16). By our choice of indexing, $n \in I_{m+2}$. Let $s \in I_m$. Each $y \in A_n$ satisfying the hypothesis of Claim A is a unique expression element in $Y + A_n$, of which there is at least one. Thus, since $Y + A_n$ is H-periodic with $|Y + A_n| = |Y| + |A_n| - 1$ by case hypothesis, we can apply the Kemperman Structure

Theorem directly to $Y + A_n$ to conclude that there are H-quasi-periodic decompositions (cf. [35, Comment c.14]) $Y = Y_1 \cup Y_0$ and $A_n = X_1 \cup X_0$ with $|Y_0| + |X_0| = |H| + 1$. Moreover, in view of [37, Theorem 5.1] and H nontrivial, either all unique expression elements are contained in $Y_0 + X_0$, or $|X_0| = 1$ with all unique expression elements involving the unique element in X_0, or $|Y_0| = 1$ with all unique expression elements involving the unique element in Y_0. If $|X_0| = 1$, then $|Y_0| + |X_0| = |H| + 1$ ensures $|Y_0| = |H| \geq 2$, in which case all unique expression elements in $Y + A_n$ must involve the unique element from X_0. However, this contradicts that there is an element $y \in A_n$ satisfying Claim A, which is part of a unique expression element in $Y + A_n$ but not the unique element from its H-coset. Therefore $|X_0| \geq 2$, ensuring that $|(y + H) \cap A_n| \geq 2$ for all $y \in A_n$. Since any element from $A_n \setminus Z$ is the unique element from its H-coset in A_n, it follows that $A_n \subseteq Z$. Repeating the above argument for an arbitrary A_k with $k \in I_{m+2}$ (using an appropriate re-indexing), we conclude that $|(y+H) \cap A_k| \geq 2$ for all $y \in A_k$, and that $A_k \subseteq Z$, which completes the claim. □

Let $s \in I_m$ be arbitrary. Recall $n \in I_{m+2}$ by our choice of indexing. In view of Claim F, any element $y \in A_n \setminus A_s$ satisfies the hypotheses of Claim A (this is trivially true if $H = G$). Thus, since Claim A ensures that $A_s \nsubseteq A_n$, we conclude that there are $|A_n \setminus A_s| \geq 3$ elements satisfying the hypotheses of Claim A. Each such $y \in A_n \setminus A_s$ is part of a unique expression element in $Y + A_n$ by Claim A. As there are at least three such y, and since $|Y + A_n| = |Y| + |A_n| - 1$ by case hypothesis, the Kemperman Structure Theorem [37, Theorem 9.1] [35, Proposition 2.2] ensures this is only possible if there are K-quasi-periodic decompositions $Y = (Y \setminus \{y\}) \cup \{y\}$ and $A_n = (A_n \setminus A_\emptyset) \cup A_\emptyset$ with $y + A_\emptyset \subseteq Y + A_n$ the subset of all unique expression elements in $Y + A_n$, and

$$K = \langle A_\emptyset - A_\emptyset \rangle.$$

In particular, $A_n \setminus A_s \subseteq A_\emptyset$ with each $x \in A_\emptyset$ being part of exactly one unique expression element $y + x \in Y + A_n$. Moreover, since $A_\emptyset$ is a subset of a K-coset, we have $|K| \geq |A_\emptyset| \geq |A_n \setminus A_s| \geq 3$. Consequently, in view of the case hypothesis and Lemma 2.5.3, it follows by a short inductive argument that there are $a_i \in A_i$ for $i \in [1, n-1]$ and $\beta \in X$ such that $X \setminus \{\beta\}$ and $A_i \setminus \{a_i\}$ for $i \in [1, n-1]$ are K-periodic with $\beta + a_1 + \ldots + a_{n-1} = y$.

Let $x_1, \ldots, x_r \in A_n \setminus A_s \subseteq A_\emptyset$ be the $r \geq 3$ distinct elements in $A_n \setminus A_s$. By translating all terms of S appropriately, we can w.l.o.g. assume $a_s = 0$. Consider an arbitrary $x_t \in A_n \setminus A_s \subseteq A_\emptyset$. By the above work, $Y + (A_n \setminus \{x_t\}) = V \setminus \{y + x_t\}$. If $V \setminus \{y + x_t\}$ is aperiodic, then Lemma 2.6 together with Kneser's Theorem implies that $|X + (A_s \cup \{x_t\}) + \sum_{\substack{i=1 \\ i \neq s}}^{n-1} A_i + (A_n \setminus \{x_t\})| \geq$ $|A_s \cup \{x_t\}| + |X + \sum_{\substack{i=1 \\ i \neq s}}^{n-1} A_i + (A_n \setminus \{x_t\})| - 1 \geq |X| + \sum_{i=1}^{n} |A_i| - n$, in which case moving x_t from A_n to A_s yields a new setpartition satisfying (14) and (15) (in view of Claim F), thus contradicting the minimality of (16) for $\mathcal{A}$. Therefore we instead conclude $H_t := \mathsf{H}(V \setminus \{y + x_t\})$ is nontrivial for every $x_t \in A_n \setminus A_s \subseteq A_\emptyset$. Since there are at least two such elements, [35, Proposition 2.1] implies the H_t are distinct cardinality two subgroups for $t \in A_n \setminus A_s$. Moreover, $V \cup \{\alpha\}$ is $(H_1 + \ldots + H_r)$-periodic for the unique element

$$\alpha \in (y + x_t + H_t) \setminus \{y + x_t\}. \tag{34}$$

Since $V \setminus \{y + x_t\}$ is periodic, [35, Comment c.6] implies

$$V = X + \sum_{i=1}^{n} A_i \quad \text{is aperiodic.} \tag{35}$$

Thus, since H is nontrivial (as noted after (16)), we conclude that $H \neq \mathsf{H}(X + \sum_{i=1}^{n} A_i)$, leaving us in the situation where $H = Z = G$ with I_e empty.

We must have $\phi_K(\beta) + \sum_{i=1}^{n} \phi_K(a_i) \in \phi_K(X) + \sum_{i=1}^{n} \phi_K(A_i)$ a unique expression element, where $a_n \in A_\emptyset$, for otherwise $V = X + \sum_{i=1}^{n} A_i$ will be K-periodic (as $X \setminus \{\beta\}$, $A_n \setminus A_\emptyset$ and $A_i \setminus \{a_i\}$ for $i \in [1, n-1]$ are all K-periodic), contrary to (35). In particular,

$$V = X + \sum_{i=1}^{n} A_i = Z_{[1,n]} \cup (\beta + \sum_{i=1}^{n-1} a_i + A_\emptyset) = Z_{[1,n]} \cup (y + A_\emptyset)$$

for some K-periodic set $Z_{[1,n]} \subseteq G$. Since $X + \sum_{i=1}^{n-1} A_i + (A_n \setminus \{x_t\}) = Z_{[1,n]} \cup (y + A_\emptyset \setminus \{x_t\})$ is a K-quasi-periodic decomposition with $H_t = \mathsf{H}\Big(Z_{[1,n]} \cup (y + A_\emptyset \setminus \{x_t\})\Big)$ and $y + A_\emptyset \setminus \{x_t\}$ a nonempty, proper subset of a K-coset, we must also have (by (12))

$$H_t = \mathsf{H}(A_\emptyset \setminus \{x_t\}) \leq K \quad \text{for any } x_t \in A_n \setminus A_s. \tag{36}$$

As a result, since $X \setminus \{\beta\}$ and $A_i \setminus \{a_i\}$ are K-periodic, it follows that $|X + H_t| = |X| + 1$ and $|A_i + H_t| = |A_i| + 1$ for all $i \in [1, n-1]$. Moreover, $A_n \setminus \{x_t\}$ is H_t-periodic. Now $0 = a_s \in A_s$ with $A_s \setminus \{0\}$ H_t-periodic. Hence, if $\{x_t, a_s\} = \{x_t, 0\} \neq H_t$, then $|(A_s \cup \{x_t\}) + H_t| = |A_s| + 3$. In such case, Lemma 2.6 together with Kneser's Theorem implies $|X + (A_s \cup \{x_t\}) + \sum_{\substack{i=1\\ i\neq s}}^{n-1} A_i + (A_n \setminus \{x_t\})| \geq$

$$|X + H_t| + |(A_s \cup \{x_t\}) + H_t| + \sum_{\substack{i=1\\ i\neq s}}^{n-1} |A_i + H_t| + |(A_n \setminus \{x_t\}) + H_t| - n|H_t| = |X| + \sum_{i=1}^{n} |A_i| - n + 1,$$

in which case moving x_t from A_n to A_s yields a new setpartition contradicting the minimality of (16) for $\mathcal{A}$. Therefore we instead conclude that

$$H_t = \{0, x_t\} \leq K, \qquad \text{for each } x_t \in A_n \setminus A_s,$$

is a cardinality two subgroup. Repeating the above arguments using any $i \in I_m$ in place of s, we find that $a_i = 0$ for all $i \in I_m$ (as $\{a_i, x_t\}$ must equal a single H_t-coset with $x_t \in H_t$ the unique nonzero element of H_t). Since $H_t = \{0, x_t\}$, it follows from (34) that

$$\alpha = y + 2x_t = y.$$

Thus $V \cup \{y\} = Z_{[1,n]} \cup \Big(y + (A_\emptyset \cup \{0\})\Big)$ is $(H_1 + \ldots + H_r)$-periodic with $H_1 + \ldots + H_r \leq K$, ensuring that $A_\emptyset \cup \{0\}$ is also $(H_1 + \ldots + H_r)$-periodic.

Suppose $|A_n| \geq m + 3$ and let $K' = H_1 + H_2 = \{0, x_1, x_2, x_1 + x_2\} \leq K$. As just noted, $A_\emptyset \cup \{0\}$ is $(H_1 + \ldots + H_r)$-periodic, and thus also K'-periodic with $K' \leq K$. Consequently, $A_n \setminus \{x_1, x_2\} = Z' \cup \{x_1 + x_2\}$ with $Z' := A_n \setminus K'$ a K'-periodic set and $x_1 + x_2$ the unique element from its K'-coset in $A_n \setminus \{x_1, x_2\}$. The sets $X \setminus \{\beta\}$ and $A_i \setminus \{a_i\}$ for $i \in [1, n-1]$ are all K-periodic, and thus also K'-periodic. The set $\Big(A_n \setminus \{x_1, x_2\}\Big) \setminus \{x_1 + x_2\} = Z' = A_n \setminus K'$ is also K'-periodic with $\phi_{K'}(x_1 + x_2) = 0$. It follows that $\phi_{K'}(y) = \phi_{K'}(\beta) + \sum_{i=1}^{n-1} \phi_{K'}(a_i) \in \phi_{K'}(X) + \sum_{i=1}^{n} \phi_{K'}(A_i)$ must be a unique expression element, as otherwise $X + \sum_{i=1}^{n} A_i$ would be K'-periodic, contradicting (35), and thus $X + (A_s \cup \{x_1, x_2\}) + \sum_{\substack{i=1 \\ i \neq s}}^{n-1} A_i + (A_n \setminus \{x_1, x_2\}) = V \setminus (y + K') \cup (y + \{0, x_1, x_2\} + \{x_1 + x_2\}) = V$ Removing x_1 and x_2 from A_n and placing them in A_s now yields a new setpartition with the same cardinality sumset as $\mathcal{A}$, contradicting the minimality of (16) for $\mathcal{A}$ (as $|A_n| \geq m + 3$). So we conclude that $|A_n| = m + 2$. Re-indexing the A_k with $k \in I_{m+2}$ and repeating these arguments for any A_k with $k \in I_{m+2}$, we conclude that $|A_k| = m + 2$ for all $k \in I_{m+2}$.

Since $A_n \setminus A_s \subseteq A_\emptyset$, we have $A_n \setminus A_\emptyset \subseteq A_s$. Hence, since $A_n \setminus A_\emptyset$ is K-periodic and $0 \in A_s$ is the unique element from it K-coset in A_s, it follows that $A_n \setminus A_\emptyset \subseteq A_s \setminus \{0\}$. If $A_n \setminus A_\emptyset \neq A_s \setminus \{0\}$, then $A_s \setminus \{0\}$ and $A_n \setminus A_\emptyset$ being K-periodic ensures $|A_s| \geq |A_n \setminus A_\emptyset| + |K| + 1 \geq |A_n \setminus A_\emptyset| + |A_\emptyset| + 1 \geq |A_n| + 1$, which is not possible. We are left to conclude $A_n \setminus A_\emptyset = A_s \setminus \{0\}$. Thus $m - 1 = |A_s \setminus \{0\}| = |A_n \setminus A_\emptyset|$, implying $m + 2 = |A_n| = (m - 1) + |A_\emptyset|$ and $|A_\emptyset| = 3$, and since $A_n \setminus A_s \subseteq A_\emptyset$ is a set of size at least three, we conclude that $A_\emptyset = A_n \setminus A_s$ and $A_n \cap A_s = A_n \setminus A_\emptyset = A_s \setminus \{0\}$. In particular, $K = \langle A_\emptyset - A_\emptyset \rangle = H_1 + H_2 + H_3 = \langle x_1, x_2, x_3 \rangle$. Since $V \cup \{y\} = Z_{[1,n]} \cup \Big(y + (A_\emptyset \cup \{0\})\Big)$ is K-periodic, with $Z_{[1,n]}$ a K-periodic set, it follows that $\{0\} \cup A_\emptyset = \{0, x_1, x_2, x_3\} = K$ is an elementary 2-group of order 4, whence $K \cong (\mathbb{Z}/2\mathbb{Z})^2$. Repeating these arguments for any $s \in I_m$ and $k \in I_{m+2}$, it follows that there exists a K-periodic subset $W \subseteq G \setminus K$ such that

$$A_s = W \cup \{0\} \quad \text{and} \quad A_k = W \cup (K \setminus \{0\}) \quad \text{for every } s \in I_m \text{ and } k \in I_{m+2}. \tag{37}$$

Since $A_s \setminus \{0\}$ is K-periodic, we have $|A_s \setminus \{0\}| = m - 1$ divisible by $|K| = 4$. Any $j \in I_{m+1}$ also has $A_j \setminus \{a_j\}$ K-periodic, whence $m = |A_j| - 1$ is divisible by 4. Since $m - 1$ and m cannot both be divisible by 4, it follows that I_{m+1} is empty.

Suppose $|I_{m+2}| \geq 2$. Then (37) implies $A_{n-1} = A_n$. Since $A_{n-1} \setminus \{a_{n-1}\}$ is K-periodic, we have $|A_{n-1}| \equiv 1 \mod |K|$. Since $A_n \setminus A_\emptyset$ is K-periodic with $|A_\emptyset| = 3$, we have $|A_n| \equiv 3$

mod $|K|$. However, since $|K| = 4$, this contradicts that $A_{n-1} = A_n$. So we conclude that $|I_{m+2}| = 1$.

We now know $A_1 = A_2 = \ldots = A_{n-1} = W \cup \{0\}$ and $A_n = W \cup \{x_1, x_2, x_3\}$ with $W \subseteq G \setminus K$ and $X \setminus \{\beta\}$ K-periodic sets and $K = \{0, x_1, x_2, x_3\}$. If $n \geq 3$, then consider the setpartition $\mathcal{A} = A'_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A'_n$ with $A'_i = W \cup \{0\}$ for $i \in [1, n-3]$, $A'_{n-2} = W \cup \{x_1\}$, $A'_{n-1} = W \cup \{0, x_2\}$ and $A'_n = W \cup \{0, x_3\}$. Then $X + \sum_{i=1}^{n} A'_i = V \cup \{\beta\} = V + K$, so that $\mathcal{A}'$ contradicts the minimality of (16) for $\mathcal{A}$ in view of (33). Therefore $n = 2$ (as we assumed $n \geq 2$ at the very start of the proof). It is now readily checked that $\mathcal{A} = A_1 \boldsymbol{\cdot} A_2$ with $A_1 = W \cup \{x\}$ and $A_2 = W \cup (K \setminus \{x\})$, for $x \in K$, are the only setpartitions partitioning the terms of S with $|X + \sum_{i=1}^{2} A_i| \geq \min\{|X| + \sum_{i=1}^{2} |A_i| - 2, |X + \Sigma_2(S)|\} = |X| + \sum_{i=1}^{2} |A_i| - 2 = |V|$, so the original setpartition $\mathcal{A}$ from the hypotheses must have this form. As the above works shows Item 1 holds for such $\mathcal{A}$, the case and proof is complete. □

We can now proceed with the proof of Theorem 1.1.

Proof Theorem 1.1. The case when L is nontrivial follows by applying the case L trivial to $\phi_L(S') \mid \phi_L(S)$. So it suffices to handle the case when L is trivial, which we now assume. Let $\mathcal{A} = A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_n$ be a setpartition with $\mathsf{S}(\mathcal{A}) \mid S$ and $|\mathsf{S}(\mathcal{A})| = |S'|$ with $|X + \sum_{i=1}^{n} A_i|$ maximal. In view of [37, Proposition 10.1], the hypotheses $S' \mid S$ and $n \leq |S'| \leq \mathsf{h}(S')$ are equivalent to such a setpartition existing. Then $\mathcal{A}$ is a setpartition with $\mathsf{S}(\mathcal{A}) \mid S$ maximal relative to X.

Suppose $|X + \sum_{i=1}^{n} A_i| \geq |X| + \sum_{i=1}^{n} |A_i| - n = |S'| - n + |X|$. Note we trivially have $|X + \Sigma_n(S)| \geq |X + \Sigma_n(\mathsf{S}(\mathcal{A}))| \geq |X + \sum_{i=1}^{n} A_i|$. Applying Lemma 2.7 to $\mathcal{A}$ allows us to assume $\mathcal{A}$ is equitable (by replacing $\mathcal{A}$ by a modified setpartition as need be, potentially losing that $|X + \sum_{i=1}^{n} A_i|$ is maximal), yielding Item 1, unless Lemma 2.7.1 holds. Assume this is the case. By translating all terms of S appropriately, we can w.l.o.g. assume $(A_1 \cup A_2) \setminus (A_1 \cap A_2) = K$ with $0 \in A_1$ and $K \setminus \{0\} \subseteq A_2$. If there is some $x \in \mathrm{Supp}(S)$ with $x \in K$, then the setpartition $\mathcal{A}' = A'_1 \boldsymbol{\cdot} A'_2$ defined by $A'_1 = (A_1 \setminus K) \cup \{x, x_1\}$ and $A'_2 = (A_2 \setminus K) \cup \{x, x_2\}$, where $x_1, x_2 \in K \setminus \{x\}$ are distinct, is an equitable setpartition with $X + A'_1 + A'_2 = X + A_1 + A_2 + K$ and $|X + A'_1 + A'_2| = |X + A_1 + A_2| + 1$. Item 1 holds in this case. If there is some $x \in \mathrm{Supp}(S)$ with $x \notin K$ and $x \notin A_1 \cap A_2$, then (12) ensures $H = \mathsf{H}(X + A_1 + A_2 \setminus \{y\}) = \mathsf{H}(A_2 \setminus \{y\})$ with $K \setminus \{0, y\}$ an H-coset, for any $y \in K \setminus \{0\}$. In such case, the setpartition $\mathcal{A}' = A'_1 \boldsymbol{\cdot} A'_2$ defined by $A'_1 = A_1 \cup \{x\}$ and $A'_2 = A_2 \setminus \{y\}$, where $y \in K \setminus \{0\}$, is an equitable setpartition, while Lemma 2.6 and Kneser's Theorem imply $|X + A'_1 + A'_2| \geq |(A_1 \cup \{x\}) + H| + |X + (A_2 \setminus \{y\}) + H| - |H| \geq |X + H| + |(A_1 \cup \{x\}) + H| + |(A_2 \setminus \{y\}) + H| - 2|H| = |X| + |A_1| + |A_2| - 1$. Item 1 follows in this case as well. Otherwise, we have $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \boldsymbol{\cdot} S) \subseteq A_1 \cap A_2$, and the remaining conclusions needed for Item 3 to hold follow from Lemma 2.7.1.

Next instead suppose $|X+\sum_{i=1}^{n}A_i|<|X|+\sum_{i=1}^{n}|A_i|-n=|S'|-n+|X|$. Let $H=\mathsf{H}(X+\sum_{i=1}^{n}A_i)$ and $Z=\bigcap_{i=1}^{n}(A_i+H)$. In view of Lemma 2.2, we can assume $\mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]}\boldsymbol{\cdot} S)\subseteq Z$ and $|(y+H)\cap A_i|\leq 1$ for all $y\in G\setminus Z$ and $i\in[1,n]$ (by replacing $\mathcal{A}$ by a modified setpartition as need be). This allows us to apply Lemma 2.3 to add the stronger assumption that $|A_i\setminus Z|\leq 1$ for all i (by replacing $\mathcal{A}$ by a modified setpartition as need be). But now Lemma 2.4 ensures that $X+\sum_{i=1}^{n}A_i=X+\Sigma_n(\mathsf{S}(\mathcal{A}))=X+\Sigma_n(S)$. Thus $H=\mathsf{H}(X+\sum_{i=1}^{n}A_i)=\mathsf{H}(X+\Sigma_n(S))$, and we can apply Lemma 2.7. Since $|\Sigma_n(S)|=|\Sigma_n(\mathsf{S}(\mathcal{A}))|=|X+\sum_{i=1}^{n}A_i|<|X|+\sum_{i=1}^{n}|A_i|-n=|S'|-n+|X|$, Lemma 2.3.1 cannot hold. Thus Lemma 2.3.2 allows us to further assume $\mathcal{A}$ is equitable (again, by replacing $\mathcal{A}$ by a modified setpartition as need be), and Item 2 follows, completing the proof. □

3. Partitioning Results for Large n

In this section, we derive stronger results in the case our setpartition $\mathcal{A}=A_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}A_n$ satisfies $\sum_{i=1}^{n}|A_i|\leq 2n$.

Lemma 3.1. *Let G be an abelian group, let $n\geq 1$, let $X\subseteq G$ be a finite, nonempty subset, let $\mathcal{A}=A_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}A_n$ be a setpartition over G, let $H=\mathsf{H}(X+\sum_{i=1}^{n}A_i)$ and let $Z=\bigcap_{i=1}^{n}(A_i+H)$. Suppose $\sum_{i=1}^{n}|A_i|\leq 2n$, $|A_i\setminus Z|\leq 1$ for all $i\in[1,n]$, and*

$$|X+\sum_{i=1}^{n}A_i|<|X+H|+\Big(\sum_{i=1}^{n}|A_i|-n\Big)|H|.$$

Then H is nontrivial and $Z=\alpha+H$ for some $\alpha\in G$.

Proof. Kneser's Theorem implies

$$|X+\sum_{i=1}^{n}A_i|\geq|X+H|+\sum_{i=1}^{n}|A_i+H|-n|H|. \tag{38}$$

If $Z=\emptyset$, then the hypothesis $|A_i\setminus Z|\leq 1$ implies $|A_i+H|=|H||A_i|=|H|$ for all i. Thus (38) implies $|X+\sum_{i=1}^{n}A_i|\geq|X+H|+\Big(\sum_{i=1}^{n}|A_i|-n\Big)|H|$, contrary to hypothesis. Therefore Z is nonempty. If H is trivial, then (38) implies $|X+\sum_{i=1}^{n}A_i|\geq|X|+\sum_{i=1}^{n}|A_i|-n=|X+H|+\Big(\sum_{i=1}^{n}|A_i|-n\Big)|H|$, contrary to hypothesis. Therefore H is nontrivial. Note that Z is H-periodic by its definition. If $|Z|\geq 2|H|$, then $|A_i+H|\geq 2|H|$ for all i, so that (38) and the hypothesis

$|S| \leq 2n$ imply

$$|X + \sum_{i=1}^{n} A_i| \geq |X + H| + n|H| \geq |X + H| + \Big(\sum_{i=1}^{n} |A_i| - n\Big)|H|,$$

contrary to hypothesis. Therefore $|Z| = |H|$, completing the proof. □

We now derive our strengthening of Theorem 1.1 for large n, mirroring the main result from [38] (which obtained the same conclusion assuming n is large with respect to the exponent).

Theorem 3.2. *Let G be an abelian group, let $n \geq 1$, let $X \subseteq G$ be a finite, nonempty subset, let $L \leq \mathsf{H}(X)$, let $S \in \mathcal{F}(G)$ be a sequence, and let $S' \mid S$ be a subsequence with $\mathsf{h}(\phi_L(S')) \leq n \leq |S'|$. Suppose $|S'| \leq 2n$. Then one of the following holds:*

1. *$n = 2$, $|S'| = |S| = |\operatorname{Supp}(\phi_L(S))|$, $\operatorname{Supp}(\phi_L(S)) = \alpha + K/L$ for some $K \leq G$ and $\alpha \in G$ with $L \leq K$ and $K/L \cong (\mathbb{Z}/2\mathbb{Z})^2$, $X \setminus (\beta + L)$ is K-periodic (or empty) for some $\beta \in X$, and $X + \Sigma_n(S) = X + (K \setminus L) + 2\alpha$ with $|X + \Sigma_n(S)| = |X| + 2|L| = (|S| - n)|L| + |X|$.*
2. *There exists an* equitable *setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$, $|\mathsf{S}(\mathcal{A})| = |S'|$, $|\phi_L(A_i)| = |A_i|$ for all $i \in [1, n]$, and $|X + \Sigma_n(S)| \geq |X + \sum_{i=1}^{n} A_i| \geq (|S'| - n)|L| + |X|$.*
3. *There exists an* equitable *setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$, $|\mathsf{S}(\mathcal{A})| = |S'|$ and $|\phi_L(A_i)| = |A_i|$ for all $i \in [1, n]$, a subgroup $K \leq H = \mathsf{H}(X + \Sigma_n(S))$ with $L < K$ proper, and $\alpha \in G$ such that*
   (a) $X + \Sigma_n(S) = X + \sum_{i=1}^{n} A_i$,
   (b) $\operatorname{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S) \subseteq \alpha + K = \bigcap_{i=1}^{n}(A_i + K)$ *and* $|A_i \setminus (\alpha + K)| \leq 1$ *for all* i,
   (c) $|X + \Sigma_n(S)| \geq |X + H| + |S_{G\setminus(\alpha+H)}| \cdot |H|$ *and* $|X + \Sigma_n(S)| \geq |X + K| + |S_{G\setminus(\alpha+K)}| \cdot |K|$,
   (d) $L + \sum_{i \in I_K} A_i = \alpha|I_K| + K$, *where* $I_K \subseteq [1, n]$ *is the nonempty subset of all* $i \in [1, n]$ *with* $A_i \subseteq \alpha + K$.

Proof. As with the proof of Theorem 1.1, it suffices to prove the case when L is trivial, as we can then apply this case to $\phi_L(S') \mid \phi_L(S)$. We divide the proof into two mains cases.

CASE 1: $|X + \Sigma_n(S)| < |S'| - n + |X|$.

We will show Item 3 holds. In this case, let $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ be an arbitrary setpartition resulting from the application of Theorem 1.1.2 to $S' \mid S$. By Theorem 1.1.2, $\mathcal{A}$ is equitable, so $|A_i| \leq 2$ for all i (as $|S'| \leq 2n$), $\mathsf{S}(\mathcal{A}) \mid S$, $|\mathsf{S}(\mathcal{A})| = |S'|$, (a) holds, $\operatorname{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S) \subseteq Z$, and $|A_i \setminus Z| \leq 1$ for all i, where $Z = \bigcap_{i=1}^{n}(A_i + H)$ and $H = \mathsf{H}(X + \Sigma_n(S))$. By case hypothesis, we have $|X + \sum_{i=1}^{n} A_i| = |X + \Sigma_n(S)| < |X| + \sum_{i=1}^{n} |A_i| - n \leq |X| + \Big(\sum_{i=1}^{n} |A_i| - n\Big)|H|$, so that Lemma 3.1 implies H is nontrivial and

$$Z = \alpha + H \quad \text{for some } \alpha \in G.$$

But now Theorem 1.1.2 implies (b) holds with $K = H$, in which case $n + |S_{G\setminus(\alpha+H)}| = \sum_{i=1}^{n}|\phi_H(A_i)|$, and now (c) holds with $K = H$ by Kneser's Theorem. Since H is nontrivial, the case when G is trivial is complete, allowing us to proceed by induction on $|G| \in \mathbb{N}\cup\{\infty\}$.

Let $I_H \subseteq [1, n]$ be all those indices $i \in [1, n]$ with $A_i \subseteq \alpha + H$, and let $I'_H \subseteq I_H$ all those indices $i \in I_H$ with $|A_i| = 1$. If $I_H = [1, n]$, then $\sum_{i=1}^{n} A_i = \sum_{i\in I_H} A_i = \alpha|I_H| + H$ and (d) holds with $K = H$, yielding Item 3 with $K = H$, as desired. Therefore, we may assume $I_H \subset [1, n]$ is a proper subset. Since (b) and (c) hold with $K = H$, it follows that

$$|X+\Sigma_n(S)| \geq |X+H|+|S_{G\setminus(\alpha+H)}|\cdot|H| = |X+H|+(n-|I_H|)|H| \geq |X|+(n-|I_H|)|H|. \tag{39}$$

Since $\mathcal{A}$ is equitable with $|S'| \leq 2n$, we have $|A_i| \leq 2$ of all i, and thus $|S'| \leq 2n - |I'_H|$. Hence the case hypothesis yields $|X + \Sigma_n(S)| \leq |S'| - n - 1 + |X| \leq n - 1 + |X| - |I'_H|$, which combines with (39) and $I_H \subset [1, n]$ proper to yield

$$|I_H \setminus I'_H| \geq (n - |I_H|)(|H| - 1) + 1 \geq |H|. \tag{40}$$

Consequently,

$$\sum_{i\in I_H}|A_i| - |I_H| + 1 = \sum_{i\in I_H\setminus I'_H}|A_i| - |I_H \setminus I'_H| + 1 \geq |H| + 1, \tag{41}$$

with the final inequality following by combining (40) with the fact that $|A_i| = 2$ for $i \in I_H \setminus I'_H$. As a result, if $|\sum_{i\in I_H} A_i| \geq \min\{|H|, \sum_{i\in I_H}|A_i| - |I_H| + 1\} = |H|$, then $|\sum_{i\in I_H} A_i| = |H|$ follows, in which case (d) holds with $K = H$, completing the proof as before. By translating all terms appropriately, we can w.l.o.g. assume $\alpha = 0$.

Let $T = S_H$, let $T' = \mathcal{S}(\prod^{\bullet}_{i\in I_H} A_i)$, let $n' = |I_H| \geq |H| > 0$, and let $H' = \mathsf{H}(\{0\}+\Sigma_{n'}(T)) \leq H$. By re-indexing the A_i, we can w.l.o.g. assume $I_H = [1, n']$. Since T' is the sequence partitioned by the setpartition $A_1 \cdot \ldots \cdot A_{n'}$, it follows that $\mathsf{h}(T') \leq n' \leq |T'|$ (see [37, Proposition 10.1]). Since the setpartition $A_1 \cdot \ldots \cdot A_n$ is equitable with $|S| \leq 2n$, we have $|A_i| \in \{1, 2\}$ for all i, so $|T'| \leq 2n'$. Since $T \in \mathcal{F}(H)$, we trivially have

$$|\Sigma_{n'}(T)| \leq |H| < \sum_{i\in I_H}|A_i| - |I_H| + 1 = |T'| - n' + 1, \tag{42}$$

with the second inequality following from (41). If $H = G$, then (a) becomes $X + \Sigma_n(S) = X + \sum_{i=1}^{n} A_i = G$ with $|S_{G\setminus(\alpha+G)}| = 0$, in which case (b)–(d) all follow trivially with $K = H = G$. Therefore we may assume $H < G$ is a proper, nontrivial subgroup, and since the stabilizer $H = \mathsf{H}(X + \Sigma_n(S))$ of a finite set must be finite, it follows that we can apply the induction hypothesis to $\{0\} + \Sigma_{n'}(T)$ using $T' \mid T$. Then Item 3 must hold for $\Sigma_{n'}(T)$ in view of (41). Let $\mathcal{B} = B_1 \cdot \ldots \cdot B_{n'}$ be the resulting setpartition and let $\beta + K$, where $K \leq H' \leq H$, be the resulting coset. Let $I_K \subseteq [1, n'] = I_H$ be the subset of indices $i \in [1, n']$ with $B_i \subseteq \beta + K$. By re-indexing the B_i, we can w.l.o.g. assume $I_K = [1, n'']$, where $n'' = |I_K|$.

Define a new setpartition $\mathcal{A}' = A'_1 \cdot \ldots \cdot A'_n$ as follows. Set $A'_i = B_i$ for $i \in [1, n']$. Since each A_i with $i \in [1, n] \setminus I_H = [n'+1, n]$ contains a term from outside $H = \alpha + H$, it follows that $|A_i| = 2$ and $|H \cap A_i| = |A_i| - 1 = 1$ for all $i \notin I_H$. Since $\sum_{i=1}^{n'} |B_i| = \sum_{i=1}^{n'} |A_i| = \sum_{i=1}^{n} |A_i \cap H|$, we have $|\mathsf{S}(\mathcal{B})^{[-1]} \cdot S_H| \geq \sum_{i=n'+1}^{n} |A_i \cap H|$. This means we can take each set A_i with $i \notin I_H$ and replace the element from $A_i \cap H$ with a separate term from $\mathsf{S}(\mathcal{B})^{[-1]} \cdot S_H$ to yield the set A'_i. As $|A_i \cap H| = 1$ for all $i \notin I_H$, we are guaranteed that $|A'_i| = |A_i|$ for all i.

By translating all terms of S by $-\beta \in H$, we can w.l.o.g. assume $\beta = 0$. Since all terms of $\mathsf{S}(\mathcal{B})^{[-1]} \cdot S_H$ are from $K = \beta + K$ by (b) (holding for $\mathcal{B}$), it follows that each A'_i, with $i > n'$, has $|A'_i \setminus K| = 1$. Since (b) holds for $\mathcal{B}$, we also have $|A'_i \setminus K| = |B_i \setminus K| \leq 1$ for all $i \leq n'$ with $K = \bigcap_{i=1}^{n} (A_i + K)$. Thus, since $\sum_{i=1}^{n'} B_i = \sum_{i=1}^{n'} A'_i$ is K-periodic, Lemma 2.4.1 implies $\sum_{i=1}^{n} A'_i = \Sigma_n(S) = \sum_{i=1}^{n} A_i$. Hence (a) holds for $\mathcal{A}' = A'_1 \cdot \ldots \cdot A'_n$. If $\sum_{i=1}^{n'} B_i = \sum_{i=1}^{n'} A'_i = H$, then (a)–(d) all hold for $\mathcal{A}'$ with $K = H$, completing the proof. Therefore we may assume $|\sum_{i=1}^{n'} B_i| \leq |H| - |K|$ (as $\sum_{i=1}^{n'} B_i \subseteq H$ is K-periodic). Thus (c) for $\mathcal{B}$ ensures that $|T_{H \setminus K}| \leq |H/K| - 2$. The first part of (c) was already established. If the second fails for $\mathcal{A}'$, then it follows that

$$|X+H| + |S_{G \setminus H}|\,|H| \leq |X + \Sigma_n(S)| < |X+K| + |S_{G \setminus K}|\,|K| \leq |X+H| + (|S_{G \setminus H}| + |H/K| - 2)|K|,$$

implying $|S_{G \setminus H}|(|H/K| - 1) \leq |H/H| - 2$, which forces $|S_{G \setminus H}| = 0$. However, in such case (a)–(d) all hold for $\mathcal{A}$ with $K = H$. Thus we can assume both parts of (c) hold for $\mathcal{A}'$ using K. In view of the construction of the A'_i and (b) for $\mathcal{B}$, it follows that (b) holds for $\mathcal{A}'$ with K, while (d) holds for $\mathcal{A}'$ with K as it holds for $\mathcal{B}$. But now (a)–(d) all hold for $\mathcal{A}'$ with subgroup $K \leq H$, which completes CASE 1.

CASE 2: $|X + \Sigma_n(S)| \geq |S'| - n + |X|$

Apply Theorem 1.1 to $S' \mid S$. If either Theorem 1.1.1 or Theorem 1.1.2 holds, then the case hypothesis ensures there exists an equitable setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$ and $|\mathsf{S}(\mathcal{A})| = |S'|$ such that

$$|X + \Sigma_n(S)| \geq |X + \sum_{i=1}^{n} A_i| \geq |S'| - n + |X|. \tag{43}$$

It follows that Item 2 holds in this case. Therefore we may instead assume Theorem 1.1.3 holds, and let $\mathcal{A} = A_1 \cdot A_2$ be the resulting setpartition. Then $|A_1| \equiv 1 \mod 4$ and $|A_2| \equiv 3 \mod 4$ with $|A_1| + |A_2| = |S'| \leq 2n = 4$. It follows that $|A_1| = 1$, $|A_2| = 3$ and $A_1 \cap A_2 = \emptyset$. Item 1 now follows from Theorem 1.1.3, completing the case and proof. $\square$

Finally, we conclude with the following application of Theorem 3.2, deriving some structural information regarding S when, in particular, $|S| = 2n$ with $|\Sigma_n(S)| \leq n+1$ and $\mathsf{h}(S) \leq n$.

Theorem 3.3. *Let G be an abelian group, let $n \geq 1$, and let $S \in \mathcal{F}(G)$ be a sequence with $|S| > n$. Suppose $|\Sigma_n(S)| \leq m+1$, where $m = \min\{n, |S|-n, |S|-\mathsf{h}(S)\}$. Then one of the following holds, with Items 1–4 only possible if $|\Sigma_n(S)| = m+1$ or $|\operatorname{Supp}(S)| = 1$.*

1. *$n = 2$, $|S| = |\operatorname{Supp}(S)|$, and $\operatorname{Supp}(S) = x + K$ for some $K \leq G$ and $x \in G$ with $K \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
2. *$m = 2$ and $\operatorname{Supp}(S) = x + K$ for some $K \leq G$ and $x \in G$ with $K \cong \mathbb{Z}/3\mathbb{Z}$.*
3. *$|\operatorname{Supp}(S)| \leq 2$.*
4. *$\operatorname{Supp}(S) \subseteq \{x-d, x, x+d\}$ for some $x, d \in G$ with $\mathsf{v}_x(S) = \mathsf{h}(S) \geq |S| - m$.*
5. *There exists $x \in G$ and a setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$, $|\mathsf{S}(\mathcal{A})| = n+m$, $\sum\limits_{i=1}^{n} A_i = \Sigma_n(S)$, $\operatorname{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S) \subseteq x + H$, $|A_i| \leq 2$ and $(x+H) \cap A_i \neq \emptyset$ for all $i \in [1,n]$, and $|\sum\limits_{i=1}^{n} A_i| = |\sum\limits_{\substack{i=1\\ i\neq j}}^{n} A_i|$ for some $j \in [1,n]$, where $H = \mathsf{H}(\Sigma_n(S))$ is nontrivial.*

Proof. If $\mathsf{h}(S) = |S|$, then $|\operatorname{Supp}(S)| = 1$, and Item 3 holds. Therefore we may assume $\mathsf{h}(S) < |S|$, and so, since $|S| > n$, may let m be the maximal integer in $[1,n]$ such that there is a subsequence $S' \mid S$ with $|S'| = n+m$ and $\mathsf{h}(S') \leq n$. If $m = n$, then $2n \leq |S|$ and $\mathsf{h}(S) \leq |S|-n$, whence $m = n = \min\{n, |S|-n, |S|-\mathsf{h}(S)\} \geq 1$. If $S' = S$, then $|S| = |S'| = n+m \leq 2n$ and $\mathsf{h}(S) \leq |S| - m = n$, whence $m = |S| - n = \min\{n, |S|-n, |S|-\mathsf{h}(S)\} \geq 1$. If $m < n$ and S' is a proper subsequence, then the maximality of m ensures that $(S')^{[-1]} \cdot S$ has only one distinct term, say x. Now $\mathsf{v}_x(S') \leq n$. If $\mathsf{v}_x(S') < n$, then $S'' = S' \cdot x$ is a subsequence with $|S''| = |S'| + 1 = n+m+1$, $\mathsf{h}(S'') \leq n$ and $m+1 \leq n$, so $m+1$ contradicts the maximality of m. Therefore $\mathsf{h}(S) = \mathsf{v}_x(S) = |S| - |S'| + n = |S| - m \geq |S| - n$ in this case, implying $\mathsf{h}(S) = |S| - |S'| + n \geq n$ and $m = |S| - \mathsf{h}(S) = \min\{n, |S|-n, |S|-\mathsf{h}(S)\} \geq 1$. In consequence, in all possible cases, we deduce that

$$m = \min\{n, |S|-n, |S|-\mathsf{h}(S)\} \geq 1. \tag{44}$$

Let $H = \mathsf{H}(\Sigma_n(S))$.

By hypothesis, $|\Sigma_n(S)| \leq m+1 = |S'| - n + 1$. If $|\Sigma_n(S)| \leq m$, then Theorem 1.1.2 applied to $S' \mid S$ (with $X = \{0\}$) yields a setpartition $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ with $\mathsf{S}(\mathcal{A}) \mid S$, $|\mathsf{S}(\mathcal{A})| = |S'| = n+m$, $\sum\limits_{i=1}^{n} A_i = \Sigma_n(S)$, $\operatorname{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S) \subseteq Z$, and $|A_i| \leq 2$ and $|A_i \setminus Z| \leq 1$ for all i, where $Z = \bigcap_{i=1}^{n}(A_i + H)$. By indexing the A_i appropriately, we can assume $|A_i| = 2$ for $i \in [1,m]$. By Lemma 3.1, H is nontrivial and $Z = x + H$ for some $x \in G$. If $|\sum\limits_{i=1}^{j} A_i| > |\sum\limits_{i=1}^{j-1} A_i|$ for all $j \in [2,m]$, then it follows that $|\Sigma_n(S)| = |\sum\limits_{i=1}^{n} A_i| \geq m+1$, contrary to assumption. Thus there is some $j \in [2, m]$ with $|\sum_{i=1}^{j} A_i| = |\sum_{i=1}^{j-1} A_i|$, meaning Item 5 holds. It remains to consider the case when $|\Sigma_n(S)| = m + 1 = |S'| - n + 1 = \sum_{i=1}^{n} |A_i| - n + 1$.

Suppose $m = 1$. If $1 = m = |S| - \mathsf{h}(S)$, then $\mathsf{h}(S) = |S| - 1$, implying $|\operatorname{Supp}(S)| \le 2$, so Item 3 follows. If $1 = m = n$, then $|\operatorname{Supp}(S)| = |\Sigma_1(S)| = |\Sigma_n(S)| \le m + 1 = 2$, and Item 3 follows. If $1 = m = |S| - n$, then $n = |S| - 1$ and $|\operatorname{Supp}(S)| = |\Sigma_1(S)| = |\sigma(S) - \Sigma_{|S|-1}(S)| = |\Sigma_n(S)| \le m + 1 = 2$, and Item 3 again follows. So we may now assume $m \ge 2$.

Apply Theorem 3.2 (with $X = \{0\}$) to $\Sigma_n(S)$ with $S' \mid S$. If Theorem 3.2.1 holds, then Item 1 follows. Otherwise, in view of $|\Sigma_n(S)| \le m + 1 = |S'| - n + 1$, let $\mathcal{A} = A_1 \cdot \ldots \cdot A_n$ be the resulting *equitable* setpartition with $Z = \bigcap_{i=1}^{n}(A_i + H)$,

$$\mathsf{S}(\mathcal{A}) \mid S, \quad |\mathsf{S}(\mathcal{A})| = |S'| = n + m, \quad \sum_{i=1}^{n} A_i = \Sigma_n(S) \quad \text{and} \quad |A_i| = 2 \text{ for all } i \in [1, m]. \tag{45}$$

CASE 1. For any setpartition $\mathcal{A}$ satisfying (45), we have $|\sum_{i=1}^{j} A_i| \ge |\sum_{i=1}^{j-1} A_i| + 1$ for all $j \in [2, m]$.

In this case, Lemma 2.1 implies $|\sum_{i=1}^{n} A_i| \ge m + 1$, with equality only possible if equality holds in each estimate $|\sum_{i=1}^{j} A_i| \ge |\sum_{i=1}^{j-1} A_i| + 1$ for $j \in [2, m]$. As this is the case, $|\sum_{i=1}^{j} A_i| = |\sum_{i=1}^{j-1} A_i| + 1$ for all $j \in [2, m]$. Moreover, this must be true under any re-indexing of the A_i with $i \in [1, m]$, whence each A_i is an arithmetic progression with a common difference $d \in G$, and each $\sum_{i=1}^{j} A_i$ is also an arithmetic progression with difference d and length $j + 1$ for $j \in [1, m]$. In particular,

$$3 \le m + 1 \le \operatorname{ord}(d),$$

and $\Sigma_n(S) = \sum_{i=1}^{n} A_i$ is an arithmetic progression with difference d, whence either H is trivial or $H = \langle d \rangle$. Thus $\sum_{\substack{i=1 \\ i \ne j}}^{n} A_i$ is aperiodic for any $j \in [1, m]$. Moreover, if $H = \langle d \rangle$, then $m = \operatorname{ord}(d) - 1$ and $\Sigma_n(S) = \sum_{i=1}^{n} A_i$ is a single H-coset, which in view of $|S| > n$ is only possible if $\operatorname{Supp}(S)$ is contained in a single H-coset.

Suppose some pair A_i and A_j are disjoint with $i, j \in [1, m]$, say $A_m = \{x, x + d\}$ and $A_{m-1} = \{y, y + d\}$. Then $y \notin \{x + d, x, x - d\}$ and

$$\Sigma_2(x \cdot (x + d) \cdot y \cdot (y + d)) = \{x + y, x + y + d, x + y + 2d, 2y + d, 2x + d\}$$

is a set of cardinality at least 4. Thus, since $A_1 + \ldots + A_{m-2} + \Sigma_2(x \cdot (x+d) \cdot y \cdot (y+d)) + A_{m+1} + \ldots + A_n \subseteq \Sigma_n(S)$ with $|\Sigma_n(S)| = m + 1$, we must have $m \ge 3$. Now $\sum_{i=1}^{m-2} A_i$ is an arithmetic progression with difference d and length $2 \le m - 1 \le \operatorname{ord}(d) - 2$, but $\{x, y\}$ is *not* an arithmetic progression with difference d since $y \notin \{x+d, x, x-d\}$. It follows that $|\sum_{i=1}^{m-2} A_i + \{x,y\}| \geq m+1$. Thus, since $|\Sigma_n(S)| = m+1$, we conclude that

$$|\Sigma_n(S)| = |\sum_{i=1}^{m-2} A_i + \{x,y\}| = |\sum_{i=1}^{m-2} A_i + \{x,y\} + \{x+d, y+d\}|, \tag{46}$$

implying that $\sum_{i=1}^{m-2} A_i + \{x,y\}$ is a translate of $\Sigma_n(S) = \sum_{i=1}^{m-2} A_i + \{x,y\} + \{x+d, y+d\} + A_{m+1} + \ldots + A_n$ with $x - y \in H = \mathsf{H}(\sum_{i=1}^{m-2} A_i + \{x,y\})$. In such case, H is nontrivial as $x \neq y$, so we have $m = \mathrm{ord}(d) - 1$ and $\mathrm{Supp}(S) \subseteq x + H = x + \langle d \rangle$ by the observation at the end of the previous paragraph. Letting $\mathcal{A}' = A'_1 \cdot \ldots \cdot A'_n$, where $A'_{m-1} = \{x,y\}$, $A'_m = \{x+d, y+d\}$ and $A'_i = A_i$ for $i \neq m-1, m$, it follows in view of (46) that Item 5 holds. So we can now assume $A_i \cap A_j \neq \emptyset$ for all $i, j \in [1,m]$. Thus, since each A_i is an arithmetic progression with difference d, it follows that there must be some $x \in \bigcap_{i=1}^m A_i$ (this is trivially true if $\mathrm{ord}(d) = 3$, as then $m \leq \mathrm{ord}(d) - 1 = 2$). Thus $A_1 \cup \ldots \cup A_m \subseteq \{x-d, x, x+d\}$ with $x \in A_i$ for all $i \in [1,m]$.

If there is some $y \in \mathrm{Supp}(S) \setminus \{x-d, x, x+d\}$, then we can exchange the term equal to $x \pm d$ in A_m with y, resulting in a set $A'_m = \{x,y\}$ that is not an arithmetic progression with difference d, while A_1 remains an arithmetic progression with difference d as $m \geq 2$. Since $\sum_{\substack{i=1\\i\neq m}}^{n} A_i$ is aperiodic, Knseser's Theorem ensures the resulting setpartition (replacing A_m by A'_m) satisfies (45), and so repeating the above arguments using the setpartition $A_1 \cdot \ldots A_{m-1} \cdot A'_m \cdot A_{m+1} \cdot \ldots \cdot A_n$ completes the proof. Therefore we may instead assume $\mathrm{Supp}(S) \subseteq \{x-d, x, x+d\}$. Indeed, we may assume $\mathrm{Supp}(S) = \{x-d, x, x+d\}$, else Item 3 holds. If $\mathrm{ord}(d) = 3$, then $2 \leq m \leq \mathrm{ord}(d) - 1$ forces $m = 2$. In this case, $\mathrm{Supp}(S) = x + K$ with $K = \{0, d, -d\}$ a subgroup of size 3, and Item 2 follows. Therefore we can assume $\mathrm{ord}(d) \geq 4$.

Suppose there is a term $y \in \mathrm{Supp}((A_1 \cdot \ldots \cdot A_m)^{[-1]} \cdot S)$ with $y \neq x$. Since $\mathrm{Supp}(S) = \{x-d, x, x+d\}$, we have $y = x \pm d$, say w.l.o.g. $y = x + d$. If $A_i = \{x, x+d\}$ for all $i \in [1,m]$, then either $|\mathrm{Supp}(S)| = 2$, yielding Item 3, or else we can exchange y for some $y' = x - d \in \mathrm{Supp}((A_1 \cdot \ldots \cdot A_m)^{[-1]} \cdot S)$. Thus, swapping y as need be, we obtain that there is some A_i with $i \in [1,m]$, say A_m, with $y \notin A_m$. Then w.l.o.g. $y = x + d$ and $A_m = \{0, x-d\}$. Note we either have $y \in \mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S)$ or $A_k = \{y\} = \{x+d\}$ for some $k > m$. Define a new setpartition $\mathcal{A}' = A'_1 \cdot \ldots \cdot A'_n$ with $A'_i = A_i$ for $i \leq m$, $A'_m = \{x-d, x+d\}$, and either $A'_i = A_i$ for all $i > m$ (if $y \in \mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S)$) or else $A'_k = \{x\}$ and $A'_i = A_i$ for all $i \in [m+1, n] \setminus \{k\}$ (if $y \notin \mathrm{Supp}(\mathsf{S}(\mathcal{A})^{[-1]} \cdot S)$). Since $\mathrm{ord}(d) \geq 4$, it follows that A'_m is not an arithmetic progression with difference d, while each A'_i with $i \in [1, m-1]$ is. Since $m \geq 2$, repeating the above arguments using the setpartition $\mathcal{A}'$ completes the proof (as $\sum_{\substack{i=1\\i\neq m}}^{n} A_i$ is aperiodic, Kneser's Theorem ensures (45) holds for $\mathcal{A}'$). So we instead assume $\mathrm{Supp}((A_1 \cdot \ldots \cdot A_m)^{[-1]} \cdot S) \subseteq \{x\}$. Combined with

$x \in A_i$ for all $i \in [1,m]$, we find $\mathsf{v}_x(S) = \mathsf{h}(S) \geq |S| - m$, and now Item 4 holds, completing CASE 1.

CASE 2. There is some setpartition $\mathcal{A}$ satisfying (45) with $|\sum_{i=1}^{n} A_i| = |\sum_{\substack{i=1\\i\neq j}}^{n} A_i|$ for some $j \in [2,m]$.

By case hypothesis and Kneser's Theorem, $H = \mathsf{H}(\Sigma_n(S))$ is nontrivial. If $m+1 = |\Sigma_n(S)| = |H|$, then $\Sigma_n(S)$ is an H-coset, which in view of $|S| > n$ is only possible if $\mathrm{Supp}(S) \subseteq x + H$ for some $x \in G$. Hence Item 5 holds in view of the case hypothesis. So we now assume $|\sum_{i=1}^{n} A_i| = |\Sigma_n(S)| \geq 2|H|$. Thus $\sum_{i=1}^{n} |\phi_H(A_i)| \geq n+1$. Let $\mathcal{B} = B_1 \cdot \ldots \cdot B_n$ be a setpartition with $\mathsf{S}(\mathcal{B}) \mid S$, $|\mathsf{S}(\mathcal{B})| = n + m$, and $\sum_{i=1}^{n} B_i = \Sigma_n(S)$ such that, letting $I_2 \subseteq [1,n]$ be the subset of all $i \in [1,n]$ with $|\phi_H(B_i)| \geq 2$, the following hold

M1. For each $i \in I_2$, there is some $b_i \in B_i$ such that $|\phi_H\big(B_i \setminus (b_i + H)\big)| = |B_i \setminus (b_i + H)|$, and
M2. either $\sum_{i \in I_2} |\phi_H(B_i)| > 2|I_2|$ or $|\phi_H(B_{i'})| = |B_{i'}|$ for some $i' \in I_2$

Since $|A_i| \leq 2$ for all i and $\sum_{i=1}^{n} |\phi_H(A_i)| \geq n+1$, $\mathcal{A}$ satisfies all these hypotheses. Let $I_1 = [1,n] \setminus I_2$ be the subset of all $i \in [1,n]$ with $|\phi_H(B_i)| = 1$, and re-index the B_i so that $I_1 = [1, |I_1|]$. Kneser's Theorem implies $|\sum_{i=1}^{n} B_i| \geq \sum_{i \in I_2} |B_i + H| - (|I_2|-1)|H| \geq \sum_{i \in I_2} |B_i| + (|I_2|+1)(|H|-1) - (|I_2|-1)|H| = \sum_{i \in I_2} |B_i| - |I_2| + (2|H| - 1)$, with the latter inequality in view of conditions M1 and M2 (note $\sum_{i \in I_2} |\phi_H(B_i)| \geq 2|I_2|$ holds trivially in view of $|\phi_H(B_i)| \geq 2$ for $i \in I_2$). Combined with the inequality $|\sum_{i=1}^{n} B_i| = |\Sigma_n(S)| \leq |S'| - n + 1 = \sum_{i=1}^{n} |B_i| - n + 1 = \sum_{i=1}^{n} |B_i| - |I_1| - |I_2| + 1$, we find

$$\sum_{i \in I_1} |B_i| \geq |I_1| + 2|H| - 2. \tag{47}$$

Consequently, I_1 is nonempty, and since we trivially have $|\sum_{i \in I_1} B_i| \leq |H|$ (as each B_i with $i \in I_1$ is contained in an H-coset), it follows that $|\sum_{i \in I_1} B_i| \leq |H| \leq \sum_{i \in I_1} |B_i| - |I_1| - (|H| - 2) < \sum_{i \in I_1} |B_i| - |I_1| + 1$, with the later inequality holding since H is nontrivial (as noted at the start of the case). Lemma 2.1 now implies there is some $j \in [2, |I_1|]$ with $|\sum_{i=1}^{j} B_i| < |\sum_{i=1}^{j-1} B_i| + |B_j| - 1$, in which case Theorem D implies

$$\sum_{i=1}^{j-1} B_i + (B_j \setminus \{y\}) = \sum_{i=1}^{j} B_i \quad \text{for all } y \in B_j. \tag{48}$$

In particular, $|B_j|\geq 2$. Also, since $|B_i|\geq 2$ for all $i\in I_2$, and since $\sum\limits_{i=1}^{n}|B_i|=|S'|\leq 2n$, it follows that

$$\sum_{i\in I_1}|B_i|\leq 2|I_1| \qquad \text{and} \qquad |I_1|\geq 2|H|-2\geq |H|, \tag{49}$$

with the latter inequality above following from the former combined with (47).

Now additionally assume that our setpartition $\mathcal{B}$ is chosen, subject to $\mathsf{S}(\mathcal{B})\mid S$, $|\mathsf{S}(\mathcal{B})|=n+m$, $\sum\limits_{i=1}^{n}B_i=\Sigma_n(S)$, M1 and M2, so that

M3. $\sum\limits_{i=1}^{n}|\phi_H(B_i)|$ is maximal.

Since $\mathcal{A}$ satisfies the defining conditions for $\mathcal{B}$, we have $\sum\limits_{i=1}^{n}|\phi_H(B_i)|\geq\sum\limits_{i=1}^{n}|\phi_H(A_i)|\geq n+1$, ensuring that I_2 is nonempty. We claim that this ensures $B_j+H\subseteq B_i$ for all $i\in[1,n]$, where $j\in I_1$ is the index defined above. Indeed, if this fails, then there is some $x\in B_j$ and $k\in[1,n]$ with $\phi_H(x)\notin\phi_H(B_k)$. In this case, remove x from B_j and place it in B_k to yield a new setpartition $\mathcal{B}=B_1'\cdot\ldots\cdot B_n'$, where $B_j'=B_j\setminus\{x\}$, $B_k'=B_k\cup\{x\}$ and $B_i'=B_i$ for $i\neq j,k$. In view of (48), we have $\mathsf{S}(\mathcal{B}')=\mathsf{S}(\mathcal{B})$, $|\mathsf{S}(\mathcal{B}')|=|\mathsf{S}(\mathcal{B})|=n+m$ and $\sum\limits_{i=1}^{n}B_i'=\Sigma_n(S)$. Since $\phi_H(x)\notin\phi_H(B_k)$, it follows that x is the unique element from its H-coset in B_k', so M1 and M2 also hold for $\mathcal{B}'$. However, since $|\phi_H(B_j)|=|\phi_H(B_j')|=1$ and $|\phi_H(B_k')|=|\phi_H(B_k)|+1$, we see that $\mathcal{B}'$ contradicts the maximality of $\sum\limits_{i=1}^{n}|\phi_H(B_i)|$ for $\mathcal{B}$ given in M3. Therefore, $B_j+H\subseteq B_i$ for all i, as claimed. Letting $x\in B_j$ and recalling that B_j is contained in an H-coset (as $j\in I_1$), it follows that $x+H=\bigcap_{i=1}^{n}(B_i+H)$. Likewise, if there were some $y\in\mathrm{Supp}(\mathsf{S}(\mathcal{B})^{[-1]}\cdot S)$ with $\phi_H(y)\neq\phi_H(x)$, then we could remove x from B_j and place y in B_j to yield a new setpartition $\mathcal{B}=B_1'\cdot\ldots\cdot B_n'$, where $B_j'=B_j\setminus\{x\}\cup\{y\}$ and $B_i'=B_i$ for $i\neq j$, which would again contradict the maximality of $\mathcal{B}$ given in M3. Therefore we may assume otherwise. In summary,

$$\mathrm{Supp}(\mathsf{S}(\mathcal{B})^{[-1]}\cdot S)\subseteq x+H=\bigcap_{i=1}^{n}(B_i+H). \tag{50}$$

Claim A. $(y+H)\cap B_i=\{y\}$ for any $i\in[1,n]$ and $y\in B_i\setminus(x+H)$.

Proof. Assume by contradiction there is some $k\in[1,n]$ and $y\in B_k\setminus(x+H)$ with $|(y+H)\cap B_k|=r\geq 2$. Since $B_i\subseteq x+H$ for each $i\in I_1$ by (50), we must have $k\in I_2$. Let $\mathcal{C}=C_1\cdot\ldots\cdot C_n$ be a setpartition with $\mathsf{S}(\mathcal{C})=\mathsf{S}(\mathcal{B})$ and $\sum\limits_{i=1}^{n}B_i=\Sigma_n(S)$ such that $C_i=B_i$ for all $i\in I_2\setminus\{k\}$, $C_k\setminus B_k\subseteq x+H$, $C_k\cap B_k=B_k\setminus\{y_1,\ldots,y_t\}$, $C_{|I_1|+1-i}\setminus(x+H)=\{y_i\}\subset C_{|I_1|+1-i}$ for $i\in[1,t]$, where $y_1,\ldots,y_t\in(y+H)\cap B_k$ are $t\in[0,r-1]$ distinct elements, and (subject to these conditions) $|(x+H)\cap C_k|$ is maximal, and then (subject to prior conditions) $t\geq 0$ is maximal.

Note $\mathcal{B}$ satisfies these conditions with $t=0$, so $\mathcal{C}$ exists. The defining conditions for $\mathcal{C}$ ensure

$$I_2' = I_2 \cup [|I_1|+1-t, |I_1|]$$

is the subset of indices $i \in [1,n]$ with $|\phi_H(C_i)| \geq 2$ and that M1 holds for all C_i with $i \in I_2' \setminus \{k\}$.

Suppose $t = r-1$. The defining conditions for $\mathcal{C}$ along with M1 for $\mathcal{B}$ ensure all elements from $C_k \setminus (\{x,y\}+H)$ are the unique element from their H-coset in C_k with $|(y+H)\cap C_k| = r-t$. Thus, since $t=r-1$, we see that M1 holds for $\mathcal{C}$. The defining conditions for $\mathcal{C}$ ensure $\phi_H(C_i) = \phi_H(B_i)$ for $i \in I_2$ and $i \in I_1 \setminus [|I_1|+1-t, |I_1|]$, while $\phi_H(B_{|I_1|+1-i}) \subset \phi_H(C_{|I_1|+1-i})$ for $i \in [1,t]$; moreover, $C_i = B_i$ for $i \in I_2 \setminus \{k\}$. Thus, since M2 holds for $\mathcal{B}$, it also holds for $\mathcal{C}$ (note $i' \neq k$ in M2 as $|(y+H)\cap B_k| \geq 2$), and $\sum_{i=1}^{n} |\phi_H(C_i)| = \sum_{i=1}^{n} |\phi_H(B_i)| + t = \sum_{i=1}^{n} |\phi_H(B_i)| + r - 1 > \sum_{i=1}^{n} |\phi_H(B_i)|$. Hence $\mathcal{C}$ contradicts the maximality condition M3 for $\mathcal{B}$. So we instead assume $t < r-1$, meaning $|(y+H)\cap C_k| = r-t \geq 2$.

Suppose $(x+H)\cap C_k = x+H$. Then $C_i \subseteq x+H \subseteq C_k$ for all $i \in [1, |I_1|-t]$ (since $\phi_H(C_i) = \phi_H(B_i)$ for all $i \notin [|I_1|+1-t, |I_1|]$). Let $y_{t+1} \in (y+H)\cap C_k$. In view of (49), we have $|I_1| \geq |H| \geq r \geq t+2$, so we can define a new setpartition $\mathcal{C}' = C_1' \cdot \ldots \cdot C_n'$, where $C_k' = C_k \setminus \{y_{t+1}\}$, $C_{|I_1|-t}' = C_{|I_1|-t} \cup \{y_{t+1}\}$, and $C_i' = C_i$ for all $i \neq k, |I_1|-t$. Then $\mathsf{S}(\mathcal{C}') = \mathsf{S}(\mathcal{C})$. We have $C_{|I_1|-t} \subseteq x+H = (x+H)\cap C_k$ and $y_{t+1} \in y+H \neq x+H$. Thus $C_{|I_1|-t} \subseteq C_k \setminus \{y_{t+1}\}$ and $C_{|I_1|-t} + C_k \subseteq (C_{|I_1|-t} \cup \{y_{t+1}\}) + (C_k \setminus \{y_{t+1}\})$, ensuring $\Sigma_n(S) = \sum_{i=1}^{n} C_i \subseteq \sum_{i=1}^{n} C_i' \subseteq \Sigma_n(S)$, forcing equality to hold. But now, since $t+1 \leq r-1$, we see that $\mathcal{C}'$ contradicts the maximality of t for $\mathcal{C}$. So we instead conclude that

$$(x+H)\cap C_k \subset x+H. \tag{51}$$

Note $\rho := |H| - |(x+H)\cap C_k| \geq 1$ by (51). Since $C_k \setminus B_k \subseteq x+H$ and $C_k \cap B_k = B_k \setminus \{y_1, \ldots, y_t\}$ with $t < r$, it follows from M1 for $\mathcal{B}$ that

$$|(C_k+H)\setminus C_k| \geq (|\phi_H(C_k)|-2)(|H|-1) + t + \rho \geq t+1. \tag{52}$$

We also have $|(C_i+H)\setminus C_i| \geq (|\phi_H(C_i)|-1)(|H|-1) \geq |H|-1$ for $i \in I_2 \setminus \{k\}$ (by M1 for $\mathcal{B}$), either $|\phi_H(C_k)| \geq 3$ (improving the final estimate in (52) by $|H|-1$) or $|(C_i+H)\setminus C_i| \geq 2(|H|-1)$ for some $i \in I_2 \setminus \{k\}$ (by M2 for $\mathcal{B}$, noting that $i' \neq k$ in view of $|(y+H)\cap B_k| \geq 2$), and $|(C_i+H)\setminus C_i| = |H|-1$ for $i \in [|I_1|+1-t, |I_1|]$. As a result,

$$\sum_{i \in I_2'} |(C_i+H)\setminus C_i| \geq \sum_{i \in I_2'} |C_i| + |I_2'|(|H|-1) + t + 1.$$

Combining this estimate with Kneser's Theorem, we obtain $|\sum_{i=1}^{n} C_i| \geq \sum_{i \in I_2'} |C_i+H| - (|I_2'|-1)|H| \geq \sum_{i \in I_2'} |C_i| - |I_2'| + t + 1 + |H|$. Combined with the inequality $|\sum_{i=1}^{n} C_i| = |\Sigma_n(S)| \leq |S'| - n + 1 =$

$\sum_{i=1}^{n}|C_i|-n+1=\sum_{i=1}^{n}|C_i|-|I_1'|-|I_2'|+1$, where $I_1':=[1,n]\setminus I_2'=[1,|I_1|-t]$, we find

$$\sum_{i\in I_1'}|C_i|\geq |I_1'|+|H|+t\geq |I_1'|+|H|. \tag{53}$$

Consequently, since we trivially have $|\sum_{i\in I_1'}C_i|\leq |H|$ (as each C_i with $i\in I_1'$ is contained in an H-coset), it follows that $|\sum_{i\in I_1'}C_i|\leq |H|<\sum_{i\in I_1'}|C_i|-|I_1'|+1$. As before, this ensures via Lemma 2.1 and Theorem D that there is some $j'\in[2,|I_1'|]$ with

$$\sum_{i=1}^{j'-1}C_i+(C_{j'}\setminus\{z\})=\sum_{i=1}^{j}C_i \quad \text{for all } z\in C_{j'}. \tag{54}$$

Suppose $C_{j'}\subseteq C_k$. Let $y_{t+1}\in(y+H)\cap C_k$. In view of (49), we have $|I_1|\geq|H|\geq r\geq t+2$, so we can define a new setpartition $\mathcal{C}'=C_1'\cdot\ldots\cdot C_n'$, where $C_k'=C_k\setminus\{y_{t+1}\}$, $C_{j'}'=C_{j'}\cup\{y_{t+1}\}$, and $C_i'=C_i$ for all $i\neq k,j'$. Then $\mathsf{S}(\mathcal{C}')=\mathsf{S}(\mathcal{C})$. We have $C_{j'}\subseteq C_k$, and thus $C_{j'}\subseteq C_k\setminus\{y_{t+1}\}$ (since $j'\in I_1'$ ensures $C_{j'}\subseteq x+H\neq y+H$ and $y_{t+1}\in y+H$). Hence $C_{j'}+C_k\subseteq(C_{j'}\cup\{y_{t+1}\})+(C_k\setminus\{y_{t+1}\})$, ensuring $\Sigma_n(S)=\sum_{i=1}^{n}C_i\subseteq\sum_{i=1}^{n}C_i'\subseteq\Sigma_n(S)$, forcing equality to hold. But now, since $t+1\leq r-1$, we see that $\mathcal{C}'$ contradicts the maximality of t for $\mathcal{C}$ (re-indexing the C_i' with $i\in I_1'$ so that $j'=|I_1'|$). So we instead conclude that $C_{j'}\nsubseteq C_k$.

Since $C_{j'}\nsubseteq C_k$ and $C_{j'}\subseteq x+H$ (as $j'\in I_1'$), there is some $z\in C_{j'}\setminus C_k$ with $z\in x+H$. Define a new setpartition $\mathcal{C}'=C_1'\cdot\ldots\cdot C_n'$, where $C_{j'}'=C_{j'}\setminus\{z\}$, $C_k'=C_k\cup\{z\}$, and $C_i'=C_i$ for all $i\neq j',k$. Then $\mathsf{S}(\mathcal{C}')=\mathsf{S}(\mathcal{C})$, and (54) ensures $\Sigma_n(S)=\sum_{i=1}^{n}C_i\subseteq\sum_{i=1}^{n}C_i'\subseteq\Sigma_n(S)$, in which case equality holds. But now $|(x+H)\cap C_k'|=|(x+H)\cap C_k|+1$, so that $\mathcal{C}'$ contradicts the maximality of $|(x+H)\cap C_k|$ for $\mathcal{C}$, completing Claim A □

Since H is nontrivial (as noted at the start of CASE 2) and $m+1=|\Sigma_n(S)|\leq|S'|-n+1=\sum_{i=1}^{n}|B_i|-n+1$, Claim A allows us to apply Lemma 2.3 to $\mathcal{B}$ (with $X=\{0\}$), giving the existence of a setpartition $\mathcal{C}=C_1\cdot\ldots\cdot C_\ell$ with $\mathsf{S}(\mathcal{C})=\mathsf{S}(\mathcal{B})$, $\sum_{i=1}^{n}C_i=\sum_{i=1}^{n}B_i=\Sigma_n(S)$, $(x+H)\subseteq Z=\bigcap_{i=1}^{n}(C_i+H)$, and $|C_i\setminus Z|\leq 1$ for all i. If $Z\neq x+H$, then $m=n$ and $|\phi_H(C_i)|=2$ for all i (recall $|S'|=n+m\leq 2n$), whence Kneser's Theorem implies $|S'|-n+1=|\Sigma_n(S)|=|\sum_{i=1}^{n}C_i|\geq(n+1)|H|\geq(|S'|-n+1)|H|$, contradicting that H is nontrivial. Therefore $Z=x+H$. It necessarily follows that $\sum_{i=1}^{n}|\phi_H(C_i)|=\sum_{i=1}^{n}|\phi_H(B_i)|$ since $x+H=\bigcap_{i=1}^{n}(B_i+H)=\bigcap_{i=1}^{n}(C_i+H)$ with $|(y+H)\cap C_i|\leq 1$ and $|(y+H)\cap B_i|\leq 1$ for all $i\in[1,n]$ and $y+H\neq x+H$ (cf. Claim A and Lemma 2.3). Applying Lemma 2.7 (with $X=\{0\}$) allows us to replace $\mathcal{C}$ with a setpartition having all the defining properties for $\mathcal{C}$ and which is equitable (Lemma 2.7.1 cannot hold since $H=\mathsf{H}(\Sigma_n(S))$ is nontrivial), so we gain that

$|C_i| \leq 2$ for all i. In doing so, we find that $\mathcal{C}$ now satisfies the defining conditions for $\mathcal{B}$. Thus we can w.l.o.g. assume the setpartition $\mathcal{B}$ defined above has $|B_i| \leq 2$ for all i. In view of (47), there are at least $2|H| - 2 \geq |H|$ sets B_i with $|B_i| = 2$ and $i \in I_1$. Since $|\sum_{i \in I_1} B_i| \leq |H|$ in view of each B_i being contained in an H-coset for $i \in I_1$, it now follows by a simple greedy algorithm [28, Proposition 2.2] that there is a subset $J_1 \subset I_1$ with $|J_1| \leq |H| - 1$ and $|\sum_{i \in J_1} B_i| = |\sum_{i \in I_1} B_i|$. Recalling (50), we find Item 5 holds using the setpartition $\mathcal{B}$, completing the case and proof. □

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