ON A CONJECTURE OF FOX-KLEITMAN AND SOME RELATED QUESTIONS

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Abstract. The Fox and Kleitman conjecture [5] regarding the maximum degree of regularity of the equation \(x_1 + \cdots + x_k - y_1 - \cdots - y_k = b_k\), as \(b_k\) runs over the positive integers, has recently been confirmed [9]. However we furnish a much simpler proof of a result establishing the right order of magnitude of the degree of regularity of the equation. Our result also gives information regarding the degree of regularity for some precise values of \(b_k\), namely \(b_k = c_{k-1} = \text{lcm}\{i : i = 1, 2, \ldots, k - 1\}\). We also consider the problem of finding the degree of regularity of some particular equations with a small number of variables.

1. Introduction

For given \(a_1, \ldots, a_k\) and \(b\) in the set \(\mathbb{Z}\) of integers, we consider the linear Diophantine equation \(L:\)

\[
\sum_{i=1}^{k} a_i x_i = b.
\]

Following [8], given \(n \in \mathbb{N}_+\), the set of positive integers, equation \(L\) is said to be \(n\)-regular if, for every \(n\)-coloring of \(\mathbb{N}_+\), there exists a monochromatic solution \(x = (x_1, \ldots, x) \in \mathbb{N}_+^k\) to \(L\).

The degree of regularity of \(L\) is the largest integer \(n \geq 0\), if any, such that \(L\) is \(n\)-regular. This (possibly infinite) number is denoted by \(\text{dor}(L)\). If \(\text{dor}(L) = \infty\), then \(L\) is said to be regular.

A well-known and challenging conjecture (known as Rado’s Boundedness Conjecture) due to Rado [8] states that there is a function \(r: \mathbb{N}_+ \to \mathbb{N}_+\) such that, given any \(n \in \mathbb{N}_+\) and any equation \(\alpha_1 x_1 + \cdots + \alpha_n x_n = 0\) with integer coefficients, if this equation is not regular over \(\mathbb{N}_+\), then it fails to be \(r(n)\)-regular. Even though there is a more general version, we state it here for a single homogeneous equation, as it has been proved by Rado [8] that if the conjecture is true for a single equation, then it is true for a system of finitely many linear equations, and as Fox and Kleitman [5] have shown, if the conjecture is true for a linear homogeneous equation, then it is true for any linear equation.

The first nontrivial case of the conjecture has been proved by Fox and Kleitman [5] by establishing the bound \(r(3) \leq 24\). In the same paper [5], the authors made the following conjecture for a very specific linear Diophantine equation.
Conjecture 1.1. Let $k \geq 1$. There exists an integer $b_k \geq 1$ such that the degree of regularity of the 2$k$-variable equation $L_k(b_k)$,
\[ x_1 + \cdots + x_k - y_1 - \cdots - y_k = b_k \]
is exactly $2k - 1$.

Fox and Kleitman [5] had proved the following.

Proposition 1.2. For any $b \in \mathbb{N}_+$, the equation $L_k(b)$ is not 2$k$-regular.

When $k = 2$, Adhikari and Eliahou [1] proved the Fox-Kleitman conjecture by establishing the following more general result:

Theorem 1.3 ([1]). For all positive integers $b$, we have
\[
\text{dor}(L_2(b)) = \begin{cases} 
1 & \text{if } b \equiv 1 \mod 2, \\
2 & \text{if } b \equiv 2, 4 \mod 6, \\
3 & \text{if } b \equiv 0 \mod 6.
\end{cases}
\]

A shorter proof of the above has been given in [2]. In [2], the following results were also established.

Theorem 1.4 ([2]). We have $\text{dor}(L_3(24)) = 4$.

Theorem 1.5 ([2]). We have $\text{dor}(L_3(120)) = 5$.

Though the full conjecture of Fox and Kleitman has been very recently established by Schoen and Taczala in [9] by generalizing a theorem of Eberhard, Green and Manners [4], in the next section, we give a very simple proof of Theorem 1.4 needing only Kneser’s Theorem.

Similarly, in Theorem 3.3 of Section 3, we give a very short proof of the fact that, writing $c_{k-1} = \text{lcm}\{i : i = 1, 2, \ldots, k-1\}$, the equation $L_k(c_{k-1})$ is $(k-1)$-regular. Our much simpler proof (which uses a result of Lev [11]), nonetheless achieves the correct order of magnitude, with a linear constant of 1 rather than the precise value 2, which is much improved as compared to earlier knowledge (as has been mentioned in [5], from a result of Strauss [10], it followed that, for an appropriate $b_k$, the equation $L_k(b_k)$ was $\Omega(\log k)$-regular). Our result also gives information regarding the degree of regularity of $L_k(b)$ with the particular value $b = c_{k-1}$, and we show that, apart from the first few values $k \leq 5$, it suffices to color the first $c_{k-1} + 1$ positive integers to find a monochromatic solution to $L(c_{k-1})$, with the solution occurring in the densest color class.

In [3], Bialostocki et al. considered equation $L$, that is $\sum_{i=1}^{k} a_i x_i = b$, where $\sum_{i=1}^{k} a_i = 0$ and $b \neq 0$. Among other things, the paper [3] established $\text{dor}(x_1 + x_2 - 2y_1 = b)$ under the condition $x_1 < y_1 < x_2$. Here in Section 4, following some line of arguments in [1], we furnish a somewhat different proof for the result on $\text{dor}(x_1 + x_2 - 2y_1 = b)$; because of Proposition 1.2, the result here is unconditional.
In what follows, for integers $a, b$ with $a \leq b$, the set of integers $x$ with $a \leq x \leq b$ will be denoted by the integer interval $[a, b]$. For a finite set $A \subseteq \mathbb{Z}$, we shall write $\text{diam} A = \max A - \min A$ to denote the diameter of $A$. Given two subsets $A$ and $B$ from an additive abelian group, we let $A + B = \{a + b : a \in A, b \in B\}$ denote their sumset and $A - B = \{a - b : a \in A, b \in B\}$ denotes their difference set. If $n \geq 0$ is an integer, then $nA = A + \ldots + A$ denotes the $n$-fold iterated sumset, where $0A := \{0\}$, while $n \cdot A = \{na : a \in A\}$ denotes the dilation of $A$.

Let $G$ be an abelian group and let $A, B \subseteq G$ be nonempty subsets. We let $H(A) = \{h \in G : h + A = A\}$ denote the stabilizer of $A$, which is a subgroup of $G$. Note $H = H(A)$ is equivalent to $H$ being the maximal subgroup for which $A$ is a union of $H$-cosets. The set $A$ is called periodic if $H(A)$ is nontrivial, and otherwise is called aperiodic. We will make use of Kneser’s Theorem (see [6, Chapter 6]), which states that $|A + B| \geq |A| + |B| - |H|$ for $H = H(A + B)$. Equivalently, $|A + B| \geq |A| + |B| - 1$ when $A + B$ is aperiodic. Iterating Kneser’s Theorem gives $\left| \sum_{i=1}^{n} A_i \right| \geq \sum_{i=1}^{n} |A_i + H| - (n-1)|H|$ for $H = H(\sum_{i=1}^{n} A_i)$.

2. $\text{dor}(L_3(24)) = 4$

For the sake of completeness, we now give an expanded version of the proof of Proposition 1.2 due to Fox and Kleitman [5].

Proof. If $b$ is not a multiple of $k$, then considering the coloring given by the residue class modulo $k$, there is no monochromatic solution to the equation $L_k(b)$ and the equation not even being $k$-regular, we are through.

So, we assume that $b$ is a multiple of $k$ and consider the following $2k$-coloring of $\mathbb{N}_+$:

for $1 \leq i \leq 2k$, the set of integers colored $i$ is defined to be

$$X_i = \bigcup_{j \geq 0} \left( \left( (i - 1)b/k + 1, ib/k \right] + 2bj \right).$$

Now, the set $X_i - X_i$ is independent of $i$. Since the set $k(X_1 - X_1) = \bigcup_{j \in \mathbb{Z}} (\left[ -b+k, b-k \right] + 2jb)$ is a union of translates of $[-b+k, b-k]$ by integer multiples of $2b$, it cannot contain $b$. Therefore, for any $i, 1 \leq i \leq 2k, k(X_i - X_i)$ does not contain $b$. This shows that $L_k(b)$ is not $2k$-regular. □

We proceed to prove that $\text{dor}(L_3(24)) = 4$ using Kneser’s Theorem. Since 5 does not divide 24, considering the mod 5 coloring shows that $L_3(24)$ is not 5-regular, and hence we only have to show that $L_3(24)$ is 4-regular, which in turn will follow from the result below and the pigeonhole principle. That result was first stated and proved in [2].

Theorem 2.1. For any subset $X \subset [0, 32]$ of cardinality $|X| = 9, 24 \in 3(X - X)$. 

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\textbf{Proof.} Suppose the result is not true and let \( X \subset [0, 32] \) be a counter example.

Now, writing \( S = X - X \), we have
\[ 24 \notin S + S + S, \]
which forces that none of the numbers 8, 12, 24 are in \( S \) as 0 \( \in \) \( S \).

If 4 \( \in \) \( S \), then none of the numbers 16, 20, 28, 32 are in \( S \). Therefore, 4 \( \in \) \( S \) would imply \( S \cap [0, 32] \cap 4\mathbb{Z} = \{ 0, 4 \} \). Hence, 4 \( \in \) \( S \) = \( X - X \) implies that, for all \( i = 0, 1, 2, 3 \), \[ |X \cap (4\mathbb{N} + i)| \leq 2 \]
and hence \( |X| \leq 8 \), a contradiction to our assumption.

Therefore, none of the numbers 4, 8, 12 nor 24 are in \( S \).

From the above observation, the difference between consecutive elements of \( X_i := X \cap (4\mathbb{Z} + i) \), for any \( i \in \{ 0, 3 \} \), is at least 16. Thus, if \( |X_i| \geq 3 \), then this is only possible if \( i = 0 \) and \( X_0 = \{ 0, 16, 32 \} \). Since \( |X| \geq 9 \) ensures by the pigeonhole principle that \( |X_i| \geq 3 \) for some \( i \), we must have \( |X_0| = |X_2| = |X_3| = 2 \) and \( X_0 = \{ 0, 16, 32 \} \).

Now, \( X_0 \subset X \), and therefore it follows that \( \{ 16, 32 \} \subset S \).

Since 24 = 20 + 20 - 16 = 28 + 28 - 32, it follows that 20, 28 \( \notin \) \( S \) and hence
\[ S \cap 4\mathbb{N} \cap [0, 32] = \{ 0, 16, 32 \}. \]

Therefore,
\[ X = \{ 0, 16, 32 \} \cup \{ a, a + 16 \} \cup \{ b, b + 16 \} \cup \{ c, c + 16 \}, \]
where \( a \equiv 1 \pmod{4} \), \( b \equiv 2 \pmod{4} \), \( c \equiv 3 \pmod{4} \) and \( 1 \leq a, b, c \leq 15 \).

Writing \( Y = X \cap [0, 15] \), we have \( Y = \{ 0, a, b, c \} \). Let \( A = (Y - Y) + (Y - Y) + (Y - Y) \).

Since \( Y - Y \subset [-15, 15] \), we have
\[ A \subset [-45, 45]. \]

Suppose there exists \( \alpha \in A \) with \( \alpha \equiv 8 \pmod{16} \). Since \( A = -A \), we can assume \( \alpha \in \{ 8, 24, 40 \} \).

If \( \alpha = 24 \), then \( 24 \in A \subset S + S + S \), and we are through.

If \( \alpha = (y_1 - y'_1) + (y_2 - y'_2) + (y_3 - y'_3) = 8 \) with \( y_i, y'_i \in Y \), then \( y_1 + 16 \in X \) and
\[ \alpha + 16 = (y_1 + 16 - y'_1) + (y_2 - y'_2) + (y_3 - y'_3) = 24 \in S + S + S, \]
and once again we are through.

Finally, if \( \alpha = 40 \), then observing that \( y'_1 + 16 \in X \), we have
\[ \alpha - 16 = (y_1 - (y'_1 + 16)) + (y_2 - y'_2) + (y_3 - y'_3) = 24 \in S + S + S. \]

Therefore, if we can show that \( A \) contains an element \( \equiv 8 \pmod{16} \), the theorem will be proved.
For a subset $Z \subseteq \mathbb{Z}$, let $\overline{Z} \subseteq \mathbb{Z}/16\mathbb{Z}$ denote its image modulo 16. Now, considering $Y$ modulo 16, as a subset of $\mathbb{Z}/16\mathbb{Z}$, $Y$ has 4 elements and $0 \in \overline{A}$. If $\overline{A}$ is periodic, it must contain 8 as all nontrivial subgroups of $\mathbb{Z}/16\mathbb{Z}$ contain 8.

Otherwise, $\overline{A}$ is aperiodic and hence Kneser’s Theorem (see remarks after the statement of Kneser’s theorem in Section 6.1 in [6]) implies

$$|\overline{A}| \geq 6|\overline{Y}| - 6 + 1 = 24 - 6 + 1 = 19,$$

which is not possible. \hfill \Box

3. The Equation $L_k(c_{k-1})$

Here we improve upon the result of Strauss mentioned in the introduction by establishing that, for some integer $b_k$, the degree of regularity of the equation $L_k(b_k): (x_1 - y_1) + \ldots + (x_k - y_k) = b_k$ is at least $k-1$. Specifically, we show that this holds with $b_k = c_{k-1} = \text{lcm}\{i : i = 1, 2, \ldots, k-1\}$.

The following is a basic consequence of the pigeonhole principle [Lemma 1, [12]].

**Theorem A.** Let $A \subseteq \mathbb{Z}$ be a finite set of integers with $|A| \geq 2$ and $\gcd(A-A) = 1$, let $s = \left\lfloor \frac{\text{diam} A}{|A|} - 2 \right\rfloor$ (for $|A| \geq 3$), and set $s = 1$ for $|A| = 2$. Let $h_1, h_2 \geq 0$ be integers with $h := h_1 + h_2 \geq 1$.

1. If $h \leq s$, then $|h_1 A - h_2 A| \geq \frac{h(h+1)}{2} |A| - h^2 + 1$.
2. If $h \geq s$, then $|h_1 A - h_2 A| \geq \frac{s(s+1)}{2} |A| - s^2 + 1 + (h - s) \text{diam} A$.

The following is a basic consequence of the pigeonhole principle [Lemma 1, [12]].

**Lemma 3.1.** Let $A \subseteq \mathbb{Z}$ be a finite, nonempty set of integers with $\text{diam} A \leq 2|A| - 2$. Then

$$[-(2|A| - 2 - \text{diam} A), 2|A| - 2 - \text{diam} A] \subseteq A - A.$$

Using the above, we can prove the following lemma.

**Lemma 3.2.** Let $r \geq 1$ and $n > r$ be integers. Suppose $X \subseteq \mathbb{Z}$ is a subset of integers with $|X| \geq n + 1$, $\text{diam} X \leq rn$ and $d = \gcd(X-X)$. Then

$$d\mathbb{Z} \cap [-rdn, rdn] \subseteq (r+1)X - (r+1)X.$$

**Proof.** Observing that the lemma is translation invariant, we may w.l.o.g. assume $0 = \min X$. If $r = 1$, then $X = [0, rn] = [0, n]$, in which case $(r+1)X - (r+1)X = 2X - 2X = [-2n, 2n]$, and the lemma holds. Therefore we may assume $r \geq 2$, and thus $|X| \geq n + 1 \geq r + 2 \geq 4$. Let $N = \max X = \text{diam} X \leq rn$.

Suppose $d \geq 2$. Then all elements of $X$ will be divisible by $d$ (in view of $0 \in X$). Let $X' = \frac{1}{d} \cdot X = \{x/d : x \in X\}$ and observe that $\gcd(X' - X') = 1$ with $X' \subset [0, \frac{rn}{d}] \subset [0, rn]$ and $|X'| = |X| \geq n + 1$. Consequently, if we knew the lemma held whenever $d = 1$, then we could apply this case to $X'$ to conclude that $[-rn, rn] \subseteq (r+1)X' - (r+1)X'$, implying (by multiplying everything by $d$) that $d\mathbb{Z} \cap [-rdn, rdn] \subseteq (r+1)X - (r+1)X$, as desired. So we
see that it suffices to consider the case when \( d = 1 \), i.e., when \( \gcd(X-X) = 1 \), which we now assume.

Since \( |X| \geq n + 1, N \leq rn \), and \( n > r \geq 2 \), we have

\[
1 \leq \left| \frac{N - 1}{|X| - 2} \right| \leq \left| \frac{N - 1}{n - 1} \right| \leq \frac{rn - 1}{n - 1} < r + 1.
\]

Consequently, applying Theorem A to \( X \) (using \( h = h_1 = r + 1 \) and \( h_2 = 0 \)), we find that

\[
|\{(r + 1)X\}| \geq \frac{s(s + 1)}{2} |X| - s^2 + 1 + (r + 1 - s)N.
\]

Note that \( \text{diam}((r + 1)X) = (r + 1)N, \ (s + 1)(|X| - 2) \geq N, \ N \geq s(n - 1) + 1 \) and \( r \geq s \).

Thus

\[
M := 2|\{(r + 1)X\}| - 2 - \text{diam}((r + 1)X) = 2|\{(r + 1)X\}| - 2 - (r + 1)N \\
\geq s(s + 1)(|X| - 2) + 2s + (r + 1 - 2s)N \\
\geq sN + 2s + (r + 1 - 2s)N = 2s + (r + 1 - s)N \\
\geq 2s + (r + 1 - s)(s(n - 1) + 1).
\]

The above bound is quadratic in \( s \) with the coefficient of \( s^2 \) negative (since \( n > 1 \)). The bound for \( M \) is thus minimized at a boundary value for \( s \). As a result, since \( 1 \leq s \leq r \) in view of (1), we conclude that \( M \geq rn + 2 > 0 \). Hence we can apply Lemma 3.1 using \( A = (r + 1)X \) to conclude that \( [-rn, rn] \subseteq [-M, M] \subseteq (r + 1)X - (r + 1)X \), completing the proof. \( \square \)

The least common multiple of the first \( r \) integers has been well studied. Bounds from Hong and Feng [7] give

\[
c_r := \text{lcm}\{i : i = 1, 2, \ldots, r\} \geq 2^{r-1},
\]

for instance, while the first few values are easily computed to be \( c_1 = 1 \), \( c_2 = 2 \), \( c_3 = 6 \), \( c_4 = 12 \), \( c_5 = 60 \), \( c_6 = 60 \), and \( c_7 = 420 \).

**Theorem 3.3.** Let \( k \geq 2 \) be a integer and let \( c_{k-1} := \text{lcm}\{i : i = 1, 2, \ldots, k-1\} \). Then the equation

\[
(x_1 - y_1) + \ldots + (x_k - y_k) = c_{k-1}
\]

is \((k - 1)\)-regular.

**Proof.** Let \( r = k - 1 \geq 1 \), let \( c = c_r \) for \( r \geq 5 \), let \( c = 3c_r = 3c_2 \) when \( r = 2 \), and let \( c = 2c_r \) for \( r \leq 4 \) with \( r \neq 2 \). Thus \( c_r \) is divisible by every integer from \([1, r]\) and \( n := \frac{c}{r} > r \) (in view of the basic lower bound mentioned above for \( c_r \) as well as the first few explicit values given above).

Let \( \chi : [1, c + 1] \rightarrow [1, r] \) be an arbitrary \( r \)-coloring. We will show that there is a monochromatic solution to the equation \((x_1 - y_1) + \ldots + (x_k - y_k) = c_r\), which will show the equation to be \( r \)-regular, as desired.

Observe that \([1, c + 1] = [1, rn + 1]\) with \( n = \frac{c}{r} > r \). Thus, by the pigeonhole principle, there is a monochromatic subset \( X \subseteq [1, rn + 1] \) with \( |X| \geq n + 1 \geq r + 2 \geq 3 \) and \( \text{diam} X \leq rn \).
Let \( d = \gcd(X - X) \). Then \( X \subseteq [1, rn + 1] \) is contained in an arithmetic progression with difference \( d \). However, since \( |X| \geq n + 1 \), this is only possible if \( d \in [1, r] \). Thus \( d \mid c_r \) by construction with \( c_r \leq c = rn \), ensuring that \( c_r \in d\mathbb{Z} \cap [1, rn] \). Applying Lemma 3.2 to \( X \) now yields \( c_r \in (r + 1)X - (r + 1)X = kX - kX \). Thus there are \( x_1, \ldots, x_k, y_1, \ldots, y_k \in X \) such that \( (x_1 - y_1) + \ldots + (x_k - y_k) = c_r = c_k - 1 \), and since all elements in \( X \) are monochromatic, this provides a monochromatic solution, completing the proof. \( \square \)

4. The Equation \( x_1 + x_2 - 2y_1 = b \)

As mentioned in the introduction, Bialostocki et al. [3] established \( \text{dor}(x_1 + x_2 - 2y_1 = b) \), under the condition \( x_1 < y_1 < x_2 \). Here, following the line of arguments in [1], we give a proof of the following.

**Theorem 4.1.** Consider the equation \( L'(b) \):

\[
x_1 + x_2 - 2y_1 = b.
\]

For all positive integers \( b \), we have

\[
\text{dor}(L'(b)) = \begin{cases} 
1 & \text{if } b \equiv 1 \mod 2, \\
2 & \text{if } b \equiv 2, 4 \mod 6, \\
3 & \text{if } b \equiv 0 \mod 6.
\end{cases}
\]

**Proof.** Because of Proposition 1.2, \( \text{dor}(L'(b)) \leq \text{dor}(L_2(b)) \leq 3 \). Again, since \( L'(b) \) is solvable in \( \mathbb{N}_+ \), we have \( 1 \leq \text{dor}(L'(b)) \).

Thus,

\[
1 \leq \text{dor}(L'(b)) \leq 3.
\]

The proof will be complete with the following observations.

**Observation 1.** Consider the 2-coloring of \( \mathbb{N}_+ \) given by coloring each integer according to its residue class modulo 2. Let \( (\lambda_1, \lambda_2, \lambda_3) \) be a monochromatic solution to \( L'(b) \) under this coloring.

This will imply

\[
\lambda_1 + \lambda_2 - 2\lambda_3 \equiv 0 \mod 2.
\]

Therefore, if \( b \) is odd, there cannot be a monochromatic solution in \( \mathbb{N}_+^4 \) and hence

\[
\text{dor}(L'(b)) = 1
\]

in this case.

**Observation 2.** Let \( b \) be even and write \( h = b/2 \) with \( h \in \mathbb{N}_+ \).

The following three vectors in \( \mathbb{N}_+^4 \) are solutions to \( L'(b) \):

\[
(b + 1, 1, 1),
(h + 1, h + 1, 1),
(b + 1, b + 1, h + 1).
\]
Since, for any 2-coloring of $\mathbb{N}_+$, at least two elements in the set $\{b + 1, b + 1, 1\}$ must be of the same color, at least one of the above three solutions must be monochromatic, and hence $\text{dor}(L'(b)) \geq 2$ when $b$ is even.

**Observation 3.** If $b \not\equiv 0 \mod 3$, then coloring each integer according to its residue class modulo 3 gives a coloring of $\mathbb{N}_+$ for which there cannot be any monochromatic solution to $L'(b)$, and hence $\text{dor}(L'(b)) \leq 2$ in this case.

**Observation 4.** Here we consider the case $b \equiv 0 \mod 6$. Since the sum of coefficients is zero, it is easy to see that if $L'(6)$ is proved to be 3-regular, then so is $L'(b)$.

Let $c: \mathbb{N}_+ \to \{0, 1, 2\}$ be an arbitrary 3-coloring of $\mathbb{N}_+$.

Consider the following families of special solutions to $L'(6)$ parametrized by $a \in \mathbb{N}_+$:

$$\begin{align*}
&(a + 6, a, a), \\
&(a + 5, a + 1, a), \\
&(a + 4, a + 2, a), \\
&(a + 3, a + 3, a), \\
&(a + 8, a, a + 1), \\
&(a + 1, a + 9, a + 2).
\end{align*}$$

The underlying sets for each of these solutions can be assumed to be multi-chromatic, and thus all sets from

$$\mathcal{E} = \{\{a, a+3\}, \{a, a+6\}, \{a, a+2, a+4\}, \{a, a+1, a+5\}, \{a, a+1, a+8\}, \{a+1, a+9, a+2\}\},$$

where $a$ ranges through $\mathbb{N}_+$, are multi-chromatic sets under $c$.

As just observed, the integer $a$ must be colored distinctly from both $a+3$ and $a+6$. Moreover, if $c(a+6) = c(a+3)$, then we would obtain the monochromatic solution $(a+6, a+6, a+3)$. It follows that

$$\{c(a), c(a+3), c(a+6)\} = \{0, 1, 2\} = \{c(a+3), c(a+6), c(a+9)\},$$

with the second equality following by the same argument used for the first, only replacing $a$ by $a+3$. Hence

$$c(a) = c(a+9).$$

Thus the color of an integer depends only on its residue class modulo 9. So, denoting the elements of $\mathbb{Z}/9\mathbb{Z}$ by $0, 1, \ldots, 8$ and their respective colors under $c$ by $c_0, c_1, \ldots, c_8$ (with indices modulo 9), we may depict the distribution of colors by the following table:

**Table 1.** The color table $C$

<table>
<thead>
<tr>
<th></th>
<th>$c_0$</th>
<th>$c_1$</th>
<th>$c_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_3$</td>
<td>$c_4$</td>
<td>$c_5$</td>
<td></td>
</tr>
<tr>
<td>$c_6$</td>
<td>$c_7$</td>
<td>$c_8$</td>
<td></td>
</tr>
</tbody>
</table>

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Since the sets \( \{a, a+2, a+4\} \), \( \{a, a+1, a+5\} \) and \( \{a+1, a+2, a+9\} \) belong to \( \mathcal{E} \) for all \( a \in \mathbb{N}_+ \), and are assumed to be multichromatic under \( c \), for all \( i \in \mathbb{Z}/9\mathbb{Z} \), we have

\[
\begin{align*}
|\{c_i, c_{i+2}, c_{i+4}\}| & \geq 2, \\
|\{c_i, c_{i+1}, c_{i+5}\}| & \geq 2, \\
|\{c_i, c_{i+1}, c_{i+2}\}| & \geq 2.
\end{align*}
\]

We may assume that the first column \((c_0, c_3, c_6)\) of \( C \) is equal to \((0, 1, 2)\) and the table is as follows:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(c_1)</td>
<td>(c_2)</td>
</tr>
<tr>
<td>1</td>
<td>(c_4)</td>
<td>(c_5)</td>
</tr>
<tr>
<td>2</td>
<td>(c_7)</td>
<td>(c_8)</td>
</tr>
</tbody>
</table>

The second and third columns of \( C \) being permutations of its first column, there are nine possible pairs holding the remaining two 0’s in \( C \):

\[
\begin{align*}
(c_1, c_2), & \quad (c_1, c_3), \quad (c_1, c_8); \\
(c_4, c_2), & \quad (c_4, c_3), \quad (c_4, c_8); \\
(c_7, c_2), & \quad (c_7, c_3), \quad (c_7, c_8).
\end{align*}
\]

However, recalling that \( c_0 = 0 \), we have

\[
\begin{align*}
|\{c_0, c_1, c_2\}| & \geq 2 \quad \text{by (4)}, \quad |\{c_0, c_1, c_3\}| \geq 2 \quad \text{by (3)}, \quad |\{c_8, c_0, c_1\}| \geq 2 \quad \text{by (4)}; \\
|\{c_0, c_2, c_4\}| & \geq 2 \quad \text{by (2)}, \quad |\{c_4, c_5, c_0\}| \geq 2 \quad \text{by (3)}, \quad |\{c_8, c_0, c_4\}| \geq 2 \quad \text{by (3)}; \\
|\{c_7, c_0, c_2\}| & \geq 2 \quad \text{by (2)}, \quad |\{c_5, c_7, c_0\}| \geq 2 \quad \text{by (2)}, \quad |\{c_7, c_8, c_0\}| \geq 2 \quad \text{by (4)}.
\end{align*}
\]

Hence none of the pairs from (5) can equal \((0, 0)\), contradicting that the two remaining 0’s in \( C \) must lie in one of the pairs from (5).

\[ \square \]

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