ON A CONJECTURE OF FOX-KLEITMAN AND ADDITIVE COMBINATORICS

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Abstract. The Fox and Kleitman conjecture [4] regarding the maximum degree of regularity of the equation $x_1 + \cdots + x_k - y_1 - \cdots - y_k = b_k$, as b_k runs over the positive integers, has recently been confirmed [8]. A much simpler proof of the main result proved here gives information regarding the degree of regularity for some precise values of b_k, namely $b_k = c_{k-1} = \text{lcm}\{i : i = 1, 2, \ldots, k-1\}$; in that process the result establishes the right order of magnitude of the degree of regularity of the equation. While the proof in [8] is achieved by generalizing a theorem of Eberhard, Green and Manners [3] on the sets with doubling less than 4, the proof of our main result uses a result of Lev [10] in Additive Combinatorics.

1. Introduction

For given $a_1, \ldots, a_k$ and b in the set $\mathbb{Z}$ of integers, we consider the linear Diophantine equation L:

$$\sum_{i=1}^{k} a_i x_i = b.$$

Following [7], given $n \in \mathbb{N}_+$, the set of positive integers, equation L is said to be *n-regular* if, for every n-coloring of $\mathbb{N}_+$, there exists a *monochromatic* solution $x = (x_1, \ldots, x) \in \mathbb{N}_+^k$ to L.

The *degree of regularity* of L is the largest integer $n \geq 0$, if any, such that L is n-regular. This (possibly infinite) number is denoted by $\text{dor}(L)$. If $\text{dor}(L) = \infty$, then L is said to be *regular*.

A well-known and challenging conjecture (known as *Rado's Boundedness Conjecture*) due to Rado [7] states that there is a function $r \colon \mathbb{N}_+ \to \mathbb{N}_+$ such that, given any $n \in \mathbb{N}_+$ and any equation $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ with integer coefficients, if this equation is not regular over $\mathbb{N}_+$, then it fails to be $r(n)$-regular. Even though there is a more general version, we state it here for a single homogeneous equation, as it has been proved by Rado [7] that if the conjecture is true for a single equation, then it is true for a system of finitely many linear equations, and as Fox and Kleitman [4] have shown, if the conjecture is true for a linear homogeneous equation, then it is true for any linear equation.

The first nontrivial case of the conjecture has been proved by Fox and Kleitman [4] by establishing the bound $r(3) \leq 24$. In the same paper [4], the authors made the following conjecture for a very specific linear Diophantine equation.

Conjecture 1.1. *Let $k \geq 1$. There exists an integer $b_k \geq 1$ such that the degree of regularity of the $2k$-variable equation $L_k(b_k)$,*

$$x_1 + \cdots + x_k - y_1 - \cdots - y_k \ = \ b_k$$

is exactly $2k-1$.

Fox and Kleitman [4] had proved the following.

Proposition 1.2. *For any $b \in \mathbb{N}_+$, the equation $L_k(b)$ is not $2k$-regular.*

When $k = 2$, Adhikari and Eliahou [1] proved the Fox-Kleitman conjecture by establishing the following more general result:

Theorem 1.3 ([1])**.** *For all positive integers b, we have*

$$\operatorname{dor}(L_2(b)) \ = \ \begin{cases} 1 & \text{if} \ \ b \equiv 1 \bmod 2, \\ 2 & \text{if} \ \ b \equiv 2, 4 \bmod 6, \\ 3 & \text{if} \ \ b \equiv 0 \bmod 6. \end{cases}$$

A shorter proof of the above has been given in [2].

Though the full conjecture of Fox and Kleitman has been very recently established by Schoen and Taczala in [8] by generalizing a theorem of Eberhard, Green and Manners [3], in Theorem 3.3 of Section 3, we give a very short proof of the fact that, writing $c_{k-1} = \operatorname{lcm}\{i : \ i = 1, 2, \ldots, k-1\}$, the equation $L_k(c_{k-1})$ is $(k-1)$-regular. Apart from giving a lower bound for the degree of regularity of $L_k(b_k)$ for the particular value $b_k = c_{k-1}$, our much simpler proof (which uses a result of Lev [10]), nonetheless achieves the correct order of magnitude, with a linear constant of 1 rather than the precise value 2, which is much improved as compared to earlier knowledge (as has been mentioned in [4], from a result of Strauss [9], it followed that, for an appropriate b_k, the equation $L_k(b_k)$ was $\Omega(\log k)$-regular). We also show that, apart from the first few values $k \leq 5$, it suffices to color the first $c_{k-1} + 1$ positive integers to find a monochromatic solution to $L(c_{k-1})$, with the solution occurring in the densest color class.

We now state the following result which was established in [2].

Theorem 1.4 ([2])**.** *We have* $\operatorname{dor}(L_3(24)) = 4$.

Thinking that it is worth recording, a very simple proof of Theorem 1.4 needing only Kneser's Theorem will be given in the next section.

We observe that the proof of the Fox-Kleitman conjecture by Schoen and Taczala and our proofs of Theorem 1.4 and Theorem 3.3 are by applications of results from Additive Combinatorics.

In what follows, for integers a, b with $a \leq b$, the set of integers x with $a \leq x \leq b$ will be denoted by the integer interval $[a, b]$. For a finite set $A \subseteq \mathbb{Z}$, we shall write $\operatorname{diam} A = \max A - \min A$ to denote the diameter of A. Given two subsets A and B from an additive abelian group, we let $A+B=\{a+b:\ a\in A,\, b\in B\}$ denote their sumset and $A-B=\{a-b:\ a\in A,\, b\in B\}$ denotes their difference set. If $n\geq 0$ is an integer, then $nA=\underbrace{A+\ldots+A}_{n}$ denotes the n-fold iterated sumset, where $0A:=\{0\}$, while $n\cdot A=\{na:\ a\in A\}$ denotes the dilation of A.

Let G be an abelian group and let $A,\, B\subseteq G$ be nonempty subsets. We let $\mathsf{H}(A)=\{h\in G: h+A=A\}$ denote the stabilizer of A, which is a subgroup of G. Note $H=\mathsf{H}(A)$ is equivalent to H being the maximal subgroup for which A is a union of H-cosets. The set A is called *periodic* if $\mathsf{H}(A)$ is nontrivial, and otherwise is called *aperiodic*. We will make use of Kneser's Theorem (see [5, Chapter 6]), which states that $|A+B|\geq|A+H|+|B+H|-|H|$ for $H=\mathsf{H}(A+B)$. Equivalently, $|A+B|\geq|A|+|B|-1$ when $A+B$ is aperiodic. Iterating Kneser's Theorem gives $|\sum_{i=1}^{n}A_i|\geq\sum_{i=1}^{n}|A_i+H|-(n-1)|H|$ for $H=\mathsf{H}(\sum_{i=1}^{n}A_i)$.

2. $\mathrm{dor}(L_3(24))=4$

For the sake of completeness, we now give an expanded version of the proof of Proposition 1.2 due to Fox and Kleitman [4].

Proof. If b is not a multiple of k, then considering the coloring given by the residue class modulo k, there is no monochromatic solution to the equation $L_k(b)$ and the equation not even being k-regular, we are through.

So, we assume that b is a multiple of k and consider the following $2k$-coloring of $\mathbb{N}_+$:
for $1\leq i\leq 2k$, the set of integers colored i is defined to be

$$X_i=\bigcup_{j\geq 0}\big(\big[(i-1)b/k+1,ib/k\big]+2bj\big).$$

Now, the set X_i-X_i is independent of i. Since the set $k(X_1-X_1)=\bigcup_{j\in\mathbb{Z}}([-b+k,b-k]+2jb)$ is a union of translates of $[-b+k,b-k]$ by integer multiples of $2b$, it cannot contain b. Therefore, for any i, $1\leq i\leq 2k$, $k(X_i-X_i)$ does not contain b. This shows that $L_k(b)$ is not $2k$-regular. □

We proceed to prove that $\mathrm{dor}(L_3(24))=4$ using Kneser's Theorem. Since 5 does not divide 24, considering the mod 5 coloring shows that $L_3(24)$ is not 5-regular, and hence we only have to show that $L_3(24)$ is 4-regular, which in turn will follow from the result below and the pigeonhole principle. That result was first stated and proved in [2].

Theorem 2.1. *For any subset $X\subset[0,32]$ of cardinality $|X|=9$,*

$$24\in 3(X-X).$$

Proof. Suppose the result is not true and let $X\subset[0,32]$ be a counter example.

Now, writing $S=X-X$, we have

$$24\notin S+S+S,$$

which forces that none of the numbers $8, 12, 24$ are in S as $0 \in S$.

If $4 \in S$, then none of the numbers $16, 20, 28, 32$ are in S. Therefore, $4 \in S$ would imply $S \cap [0, 32] \cap 4\mathbb{Z} = \{0, 4\}$. Hence, $4 \in S = X - X$ implies that, for all $i = 0, 1, 2, 3$,

$$|X \cap (4\mathbb{N} + i)| \leq 2$$

and hence $|X| \leq 8$, a contradiction to our assumption.

Therefore, none of the numbers 4, 8, 12 nor 24 are in S.

From the above observation, the difference between consecutive elements of $X_i := X \cap (4\mathbb{Z} + i)$, for any $i \in [0, 3]$, is at least 16. Thus, if $|X_i| \geq 3$, then this is only possible if $i = 0$ and $X_0 = \{0, 16, 32\}$. Since $|X| \geq 9$ ensures by the pigeonhole principle that $|X_i| \geq 3$ for some i, we must have $|X_1| = |X_2| = |X_3| = 2$ and $X_0 = \{0, 16, 32\}$.

Now, $X_0 \subset X$, and therefore it follows that $\{16, 32\} \subset S$.

Since $24 = 20 + 20 - 16 = 28 + 28 - 32$, it follows that $20, 28 \notin S$ and hence

$$S \cap 4\mathbb{N} \cap [0, 32] = \{0, 16, 32\}.$$

Therefore,

$$X = \{0, 16, 32\} \cup \{a, a + 16\} \cup \{b, b + 16\} \cup \{c, c + 16\},$$

where $a \equiv 1 \pmod 4$, $b \equiv 2 \pmod 4$, $c \equiv 3 \pmod 4$ and $1 \leq a, b, c \leq 15$.

Writing $Y = X \cap [0, 15]$, we have $Y = \{0, a, b, c\}$. Let $A = (Y - Y) + (Y - Y) + (Y - Y)$. Since $Y - Y \subset [-15, 15]$, we have

$$A \subset [-45, 45].$$

Suppose there exists $\alpha \in A$ with $\alpha \equiv 8 \pmod{16}$. Since $A = -A$, we can assume $\alpha \in \{8, 24, 40\}$.

If $\alpha = 24$, then $24 \in A \subset S + S + S$, and we are through.

If $\alpha = (y_1 - y_1') + (y_2 - y_2') + (y_3 - y_3') = 8$ with y_i, $y_i' \in Y$, then $y_1 + 16 \in X$ and

$$\alpha + 16 = (y_1 + 16 - y_1') + (y_2 - y_2') + (y_3 - y_3') = 24 \in S + S + S,$$

and once again we are through.

Finally, if $\alpha = 40$, then observing that $y_1' + 16 \in X$, we have

$$\alpha - 16 = (y_1 - (y_1' + 16)) + (y_2 - y_2') + (y_3 - y_3') = 24 \in S + S + S.$$

Therefore, if we can show that A contains an element $\equiv 8 \pmod{16}$, the theorem will be proved.

For a subset $Z \subseteq \mathbb{Z}$, let $\overline{Z} \subseteq \mathbb{Z}/16\mathbb{Z}$ denote its image modulo 16. Now, considering Y modulo 16, as a subset of $\mathbb{Z}/16\mathbb{Z}$, $\overline{Y}$ has 4 elements and $0 \in \overline{A}$. If $\overline{A}$ is periodic, it must contain 8 as all nontrivial subgroups of $\mathbb{Z}/16\mathbb{Z}$ contain 8.

Otherwise, $\overline{A}$ is aperiodic and hence Kneser's Theorem (see remarks after the statement of Kneser's theorem in Section 6.1 in [5]) implies

$$|\overline{A}| \geq 6|\overline{Y}| - 6 + 1 = 24 - 6 + 1 = 19,$$

which is not possible. □

3. The Equation $L_k(c_{k-1})$

Here we improve upon the result of Strauss mentioned in the introduction by establishing that, for some integer b_k, the degree of regularity of the equation $L_k(b_k)$: $(x_1 - y_1) + \ldots + (x_k - y_k) = b_k$ is at least $k-1$. Specifically, we show that this holds with $b_k = c_{k-1} = \operatorname{lcm}\{i : \ i = 1, 2, \ldots, k-1\}$.

The following is a result of Lev [Corollary, [10]]. Here, the case $h = 1$ is trivial.

Theorem A. *Let $A \subseteq \mathbb{Z}$ be a finite set of integers with $|A| \geq 2$ and $\gcd(A - A) = 1$, let $s = \lfloor \frac{\operatorname{diam} A - 1}{|A| - 2} \rfloor$ (for $|A| \geq 3$), and set $s = 1$ for $|A| = 2$. Let $h_1, h_2 \geq 0$ be integers with $h := h_1 + h_2 \geq 1$.*

1. *If $h \leq s$, then $|h_1 A - h_2 A| \geq \frac{h(h+1)}{2}|A| - h^2 + 1$.*
2. *If $h \geq s$, then $|h_1 A - h_2 A| \geq \frac{s(s+1)}{2}|A| - s^2 + 1 + (h - s) \operatorname{diam} A$.*

The following is a basic consequence of the pigeonhole principle [Lemma 1, [11]].

Lemma 3.1. *Let $A \subseteq \mathbb{Z}$ be a finite, nonempty set of integers with $\operatorname{diam} A \leq 2|A| - 2$. Then*

$$[-(2|A| - 2 - \operatorname{diam} A), 2|A| - 2 - \operatorname{diam} A] \subseteq A - A.$$

Using the above, we can prove the following lemma.

Lemma 3.2. *Let $r \geq 1$ and $n > r$ be integers. Suppose $X \subseteq \mathbb{Z}$ is a subset of integers with $|X| \geq n + 1$, $\operatorname{diam} X \leq rn$ and $d = \gcd(X - X)$. Then*

$$d\mathbb{Z} \cap [-rdn, rdn] \subseteq (r+1)X - (r+1)X.$$

Proof. Observing that the lemma is translation invariant, we may w.l.o.g. assume $0 = \min X$. If $r = 1$, then $X = [0, rn] = [0, n]$, in which case $(r+1)X - (r+1)X = 2X - 2X = [-2n, 2n]$, and the lemma holds. Therefore we may assume $r \geq 2$, and thus $|X| \geq n + 1 \geq r + 2 \geq 4$. Let $N = \max X = \operatorname{diam} X \leq rn$.

Suppose $d \geq 2$. Then all elements of X will be divisible by d (in view of $0 \in X$). Let $X' = \frac{1}{d} \cdot X = \{x/d : \ x \in X\}$ and observe that $\gcd(X' - X') = 1$ with $X' \subseteq [0, \lfloor \frac{rn}{d} \rfloor] \subseteq [0, rn]$ and $|X'| = |X| \geq n + 1$. Consequently, if we knew the lemma held whenever $d = 1$, then we could apply this case to X' to conclude that $[-rn, rn] \subseteq (r+1)X' - (r+1)X'$, implying (by multiplying everything by d) that $d\mathbb{Z} \cap [-rdn, rdn] \subseteq (r+1)X - (r+1)X$, as desired. So we see that it suffices to consider the case when $d = 1$, i.e., when $\gcd(X - X) = 1$, which we now assume.

Since $|X| \geq n+1$, $N \leq rn$, and $n > r \geq 2$, we have

$$s := \left\lfloor \frac{N-1}{|X|-2} \right\rfloor \leq \frac{N-1}{|X|-2} \leq \frac{N-1}{n-1} \leq \frac{rn-1}{n-1} < r+1. \tag{1}$$

Consequently, applying Theorem A to X (using $h = h_1 = r+1$ and $h_2 = 0$), we find that

$$|(r+1)X| \geq \frac{s(s+1)}{2}|X| - s^2 + 1 + (r+1-s)N.$$

Note that $\operatorname{diam}\big((r+1)X\big) = (r+1)N$, $(s+1)(|X|-2) \geq N$, $N \geq s(n-1)+1$ and $r \geq s$. Thus

$$\begin{aligned} M: &= 2|(r+1)X| - 2 - \operatorname{diam}\big((r+1)X\big) = 2|(r+1)X| - 2 - (r+1)N \\ &\geq s(s+1)(|X|-2) + 2s + (r+1-2s)N \\ &\geq sN + 2s + (r+1-2s)N = 2s + (r+1-s)N \\ &\geq 2s + (r+1-s)(s(n-1)+1). \end{aligned}$$

The above bound is quadratic in s with the coefficient of s^2 negative (since $n > 1$). The bound for M is thus minimized at a boundary value for s. As a result, since $1 \leq s \leq r$ in view of (1), we conclude that $M \geq rn+2 > 0$. Hence we can apply Lemma 3.1 using $A = (r+1)X$ to conclude that $[-rn, rn] \subseteq [-M, M] \subseteq (r+1)X - (r+1)X$, completing the proof. □

The least common multiple of the first r integers has been well studied. Bounds from Hong and Feng [6] give

$$c_r := \operatorname{lcm}\{i : \ i = 1, 2, \ldots, r\} \geq 2^{r-1},$$

for instance, while the first few values are easily computed to be $c_1 = 1$, $c_2 = 2$, $c_3 = 6$, $c_4 = 12$, $c_5 = 60$, $c_6 = 60$, and $c_7 = 420$.

Theorem 3.3. *Let $k \geq 2$ be a integer and let $c_{k-1} = \operatorname{lcm}\{i : \ i = 1, 2, \ldots, k-1\}$. Then the equation*

$$(x_1 - y_1) + \ldots + (x_k - y_k) = c_{k-1}$$

is $(k-1)$-regular.

Proof. Let $r = k-1 \geq 1$, let $c = c_r$ for $r \geq 5$, let $c = 3c_r = 3c_2$ when $r = 2$, and let $c = 2c_r$ for $r \leq 4$ with $r \neq 2$. Thus c_r is divisible by every integer from $[1, r]$ and $n := \frac{c}{r} > r$ (in view of the basic lower bound mentioned above for c_r as well as the first few explicit values given above). Let $\chi : [1, c+1] \to [1, r]$ be an arbitrary r-coloring. We will show that there is a monochromatic solution to the equation $(x_1 - y_1) + \ldots + (x_k - y_k) = c_r$, which will show the equation to be r-regular, as desired.

Observe that $[1, c+1] = [1, rn+1]$ with $n = \frac{c}{r} > r$. Thus, by the pigeonhole principle, there is a monochromatic subset $X \subseteq [1, rn+1]$ with $|X| \geq n+1 \geq r+2 \geq 3$ and $\operatorname{diam} X \leq rn$. Let $d = \gcd(X - X)$. Then $X \subseteq [1, rn+1]$ is contained in an arithmetic progression with difference d. However, since $|X| \geq n+1$, this is only possible if $d \in [1, r]$. Thus $d \mid c_r$ by construction with $c_r \leq c = rn$, ensuring that $c_r \in d\mathbb{Z} \cap [1, rn]$. Applying Lemma 3.2 to X now yields $c_r \in (r+1)X - (r+1)X = kX - kX$. Thus there are $x_1, \ldots, x_k, y_1, \ldots, y_k \in X$ such that $(x_1 - y_1) + \ldots + (x_k - y_k) = c_r = c_{k-1}$, and since all elements in X are monochromatic, this provides a monochromatic solution, completing the proof. □

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